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Degree Mathematics Formulae & Examples

By

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PREFACE

THE concise recapitulation of essentials in Degree Mathematics, here provided, with about 400 problems for revision work, and their answers, is in part complementary to the author's *University Mathematics* published by Blackie. It provides entirely different examples, and will also be useful to those students who rely only upon lecture notes supplemented by practice in the working out of problems.

Three extra chapters on double and triple integrals, Fourier series, and spherical trigonometry have been added to assist students taking an Engineering degree and also those taking Ancillary Mathematics for a London Special Chemistry degree.

As in *University Mathematics* all the questions are taken from London General and Engineering degree papers. The examples marked I are from Part I of the Engineering degree and those marked II from Part II of this examination, whilst the remaining questions are taken from General degree papers.

My grateful thanks are due to the Senate of the London University for their kind permission to reproduce the questions contained in this book, and also to my son Michael for his assistance with the proofs.

It is possible that some errors have been overlooked and I should be grateful to anyone bringing corrections to my notice.

JOSEPH BLAKEY.

CONSTANTINE TECHNICAL COLLEGE,
MIDDLESBROUGH, *July*, 1952.

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CHAPTER I

Revision (Intermediate)

Algebra

1. Quadratic equation $ax^2 + bx + c = 0$; roots α and β with $\alpha > \beta$.

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

$\alpha + \beta = -b/a$, $\alpha\beta = c/a$.

Roots real if $b^2 > 4ac$, coincident if $b^2 = 4ac$, imaginary if $b^2 < 4ac$, and rational if $b^2 - 4ac$ is a complete square.

2. Quadratic expression $E = ax^2 + bx + c$.

If α and β are the roots of $E = 0$ as above, then

(i) When a is positive,
E is positive if $x > \alpha$ or $< \beta$;
E is negative if $\beta < x < \alpha$.

(ii) When a is negative,
E is positive if $\beta < x < \alpha$;
E is negative if $x > \alpha$ or $< \beta$.

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3. For the arithmetical progression (A.P.)

$$a, \ a + d, \ a + 2d, \ldots$$

(i) The nth term is $a + (n - 1)d$.

(ii) Sum to n terms $= \frac{n}{2}\{2a + (n - 1)d\} = \frac{n}{2}(a + l)$, where l is the last term.

4. For the geometrical progression (G.P.)

$$a, \ ar, \ ar^2, \ ar^3, \ldots$$

(i) The nth term is ar^{n-1}.

(ii) Sum to n terms $= \frac{a(1 - r^n)}{1 - r}$.

If the numerical value of r ($|r|$ or "modulus of r") be less than unity, then the sum to infinity is $a/(1 - r)$.

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5. The first n natural numbers are 1, 2, 3, . . ., n.

If S_1 = their sum, S_2 = the sum of their squares, and S_3 = the sum of their cubes.

$$S_1 = \frac{n}{2}(n+1)\,;\ S_2 = \frac{n(n+1)(2n+1)}{6}\,;\ S_3 = \left\{\frac{n(n+1)}{2}\right\}^2.$$

6. The binomial theorem states that, for all rational values of n,

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \ldots$$

$$+ \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}a^{n-r}x^r + \ldots$$

Valid for $|x| < |a|$ if n be fractional or negative, and when $a = 1$.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$$

$$+ \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots$$

Valid for $|x| < 1$, if n be fractional or negative.

7. Trigonometry

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Angle	$90° - \theta$	$90° + \theta$	$180° - \theta$	$180° + \theta$	$270° - \theta$	$270° + \theta$	$360° - \theta$ *or* $-\theta$
sin	$+\cos\theta$	$+\cos\theta$	$+\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$
cos	$+\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$+\sin\theta$	$+\cos\theta$
tan	$+\cot\theta$	$-\cot\theta$	$-\tan\theta$	$+\tan\theta$	$+\cot\theta$	$-\cot\theta$	$-\tan\theta$

8.

Angle	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cos	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$\frac{1}{2}$	0
tan	0	$1/\sqrt{3}$	1	$\sqrt{3}$	$\pm\infty$

9. If $\alpha = \sin^{-1} a$, $\beta = \cos^{-1} b$, $\gamma = \tan^{-1} c$, where α is the angle between $-90°$ and $+90°$ whose sine has the value a, β is the angle

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between 0° and 180° whose cosine has the value b, and γ is the angle between $-90°$ and $+90°$ whose tangent has the value c, then the general solutions of $\sin\theta = a$; $\cos\theta = b$; $\tan\theta = c$ are given by

$$\theta = m \,.\, 180° + (-1)^m \alpha,$$
$$\theta = n \,.\, 360° \pm \beta,$$
$$\theta = p \,.\, 180° + \gamma,$$

respectively, where m, n, p are any integers.

10. For all values of x

$\operatorname{cosec} x = 1/\sin x$; $\sec x = 1/\cos x$; $\cot x = 1/\tan x = \cos x/\sin x$; $\sin^2 x + \cos^2 x = 1$; $\tan^2 x + 1 = \sec^2 x$; $1 + \cot^2 x = \operatorname{cosec}^2 x$.

11.
$$\sin(A + B) = \sin A \cos B + \cos A \sin B.$$
$$\sin(A - B) = \sin A \cos B - \cos A \sin B.$$
$$\cos(A + B) = \cos A \cos B - \sin A \sin B.$$
$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$
$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$
$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

12.
$$\sin 2\theta = 2 \sin\theta \cos\theta = 2 \tan\theta/(1 + \tan^2\theta)$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$
$$= \frac{1 - \tan^2\theta}{1 + \tan^2\theta}.$$
$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}.$$
$$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta); \quad \sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta).$$

13.
$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$
$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

14.
$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2};$$
$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}.$$

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$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2};$$

$$\cos x - \cos y = 2\sin\frac{x+y}{2}\sin\frac{y-x}{2},$$

$$2\sin\theta\cos\phi = \sin(\theta+\phi) + \sin(\theta-\phi);$$

$$2\cos\theta\sin\phi = \sin(\theta+\phi) - \sin(\theta-\phi);$$

$$2\cos\theta\cos\phi = \cos(\theta+\phi) + \cos(\theta-\phi);$$

$$2\sin\theta\sin\phi = \cos(\theta-\phi) - \cos(\theta+\phi).$$

15. With the standard notation for a triangle ABC:

$$\text{Sine rule } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

$$\text{Cosine rule } \cos C = \frac{a^2+b^2-c^2}{2ab}, \text{ etc.}$$

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot C/2, \text{ etc.}$$

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \text{ etc.}$$

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$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}, \text{ etc.}$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \text{ etc.}$$

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}, \text{ etc.}$$

$$\text{Area of triangle ABC} = \Delta = \tfrac{1}{2}bc\sin A = \tfrac{1}{2}ca\sin B = \tfrac{1}{2}ab\sin C$$
$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

$$\text{Radius of circumcircle} = R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}.$$

$$r = (s-a)\tan A/2 = (s-b)\tan B/2 = (s-c)\tan C/2 = \Delta/s.$$

$$r_1 = s\tan A/2 = (s-c)\cot B/2 = (s-b)\cot C/2 = \Delta/(s-a).$$

$$r_2 = (s-c)\cot A/2 = s\tan B/2 = (s-a)\cot C/2 = \Delta/(s-b).$$

$$r_3 = (s-b)\cot A/2 = (s-a)\cot B/2 = s\tan C/2 = \Delta/(s-c).$$

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Co-ordinate Geometry

16. Distance between the points (x_1, y_1) and (x_2, y_2) is given by $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Point $(\bar{x}, \bar{y})$ dividing the join of (x_1, y_1), (x_2, y_2) in the ratio $\lambda_2 : \lambda_1$ is given by

$$\bar{x} = \frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2}, \quad \bar{y} = \frac{\lambda_1 y_1 + \lambda_2 y_2}{\lambda_1 + \lambda_2},$$

where λ_1 and λ_2 have the same sign for internal division and opposite signs for external division.

Area of triangle joining the points (x_1, y_1), (x_2, y_2), (x_3, y_3) is $\frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$.

17. Equations of a straight line are :

General equation : $ax + by + c = 0$; a, b, c are constants. Slope equation : $y = mx + c$. m is the slope and c the intercept on OY.

Intercept equation : $\frac{x}{a} + \frac{y}{b} = 1$. a and b are intercepts on OX and OY respectively.

Perpendicular equation : $x \cos \alpha + y \sin \alpha = p$, where p (always $+ ve$) is the perpendicular from O on the line and makes an angle α with OX.

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Equation of line joining (x_1, y_1), (x_2, y_2) : $(y - y_1)/(y_1 - y_2) = (x - x_1)/(x_1 - x_2)$.

Equation of line, slope m through (x_1, y_1) : $y - y_1 = m(x - x_1)$.

18. Angle θ between two lines of slopes m_1, m_2 is given by

$$\theta = \tan^{-1} \pm \frac{m_1 - m_2}{1 + m_1 m_2}.$$

The lines are parallel if $m_1 = m_2$, and perpendicular if $m_1 m_2 = -1$.

19. The perpendicular from (x_1, y_1) on the line $x \cos \alpha + y \sin \alpha = p$ is of length

$$p - x_1 \cos \alpha - y_1 \sin \alpha.$$

The perpendicular from (x_1, y_1) on the line $ax + by + c = 0$ is of length

$$\pm \frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}},$$

where the $+$ sign is taken if c is $+ ve$, and the negative sign if c is $- ve$.

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The equations of the bisectors of the angles between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are given by

$$\frac{a_1x + b_1y + c_1}{\sqrt{(a_1^2 + b_1^2)}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{(a_2^2 + b_2^2)}}.$$

20. The equation of the circle, centre (x_1, y_1), radius r, is

$$(x - x_1)^2 + (y - y_1)^2 = r^2;$$

and if the origin O be its centre, the equation is

$$x^2 + y^2 = r^2.$$

The general equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0;$$

its centre is $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

The equation of the circle on the join of (x_1, y_1), (x_2, y_2) as diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

The equation of the tangent at (x_1, y_1) to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

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The equation of the tangent of slope m to the circle $x^2 + y^2 = r^2$ is

$$y = mx \pm r\sqrt{(1 + m^2)}.$$

The length (t) of the tangent from (x_1, y_1) to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

is given by

$$t^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$$

21. Roots of the general equation of the nth degree: any equation of the nth degree in x must have n roots.

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0,$$

$$\sum_1^n \alpha_r = -a_1/a_0; \quad \sum_1^n \alpha_r\alpha_p = +\frac{a_2}{a_0} (r \neq p);$$

$$\sum_1^n \alpha_r\alpha_p\alpha_q = -a_3/a_0 (r \neq p \neq q); \text{ and so on.}$$

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EXAMPLES I

1. If x be real, find the range of values of the function

$$\frac{x^2 - x + 4}{x^2 + x + 4}.$$

Find also the range of values of x for which the value of the function is greater than 4/3, and draw a rough graph of the function.

2. Show that, if α and β are the roots of the equation

$$ax^2 + 2bx + c + \lambda(a_1x^2 + 2b_1x + c_1) = 0,$$

it is possible to find constants A, B, C independent of λ, such that

$$A\alpha\beta + B(\alpha + \beta) + C = 0.$$

Determine such a set of non-zero values of A, B, and C.

3. Prove that the quadratic equations

$$a_1x^2 + b_1x + c_1 = 0, \quad a_2x^2 + b_2x + c_2 = 0$$

have a common root if, and only if,

$$(b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1) - (c_1a_2 - c_2a_1)^2 = 0.$$

If $f(x)$ be a polynomial, prove that α is a repeated root of the equation $f(x) = 0$ if, and only if, $f(\alpha) = f'(\alpha) = 0$.

Hence or otherwise, prove that the equation

$$ax^3 + bx^2 + cx + d = 0$$

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has a repeated root if, and only if,

$$(bc - 9ad)^2 - 4(b^2 - 3ac)(c^2 - 3bd) = 0.$$

4. Show that, for all real values of k other than zero, the line $y = k$ meets the curve $y(x - 3)(x + 1) = ax + b$ in real points if b lies between $-3a$ and a.

Sketch the curve in the cases :—

$$\text{(i) } a = -3,\ b = +7\ ; \quad \text{(ii) } a = +3,\ b = +7.$$

5. If a, b, c are real, find the condition that $ax^2 + bx + c$ may have the same sign for all values of x.

Show that the expression $\dfrac{2x^2 - 3ax + a^2}{x^2 - 3x + 2}$ can assume any real value for real values of x only if $1 < a < 4$.

If $a = 0$, prove that there will be two extreme values between which the value of the expression cannot lie and determine these values.

6. If $y = \dfrac{a_1x^2 + b_1x + c_1}{a_2x^2 + b_2x + c_1}$, where all the coefficients are real, show

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that the stationary values of y are the roots of the quadratic equation

$$(b_1 - b_2y)^2 - 4(a_1 - a_2y)(c_1 - c_2y) = 0.$$

If the roots of the equation $a_ix^2 + b_ix + c_i = 0$ are α_i, β_i $(i = 1, 2)$, show that the discriminant of the above quadratic in y is

$$16a_1^2a_2^2(\alpha_1 - \alpha_2)(\alpha_1 - \beta_2)(\beta_1 - \alpha_2)(\beta_1 - \beta_2).$$

Hence show that, if $\alpha_1, \beta_1, \alpha_2, \beta_2$ are all real, y has real stationary values if α_1 and β_1 lie between α_2 and β_2, or if α_2 and β_2 lie between α_1 and β_1.

(The discriminant of $ay^2 + by + c$ is $b^2 - 4ac$.)

7. If none of the quantities $(a + 1)$, $(b + 1)$, and $(c + 1)$ is zero, and if

$$y + z = ax + xyz,$$

$$z + x = by + xyz$$

and

$$x + y = cz + xyz,$$

prove that

$$(a + 1)x = (b + 1)y = (c + 1)z,$$

and hence solve the first three equations for x, y and z.

8. (i) If

$x^2 = y^2 + z^2 + 2ayz$, $y^2 = z^2 + x^2 + 2bzx$, $z^2 = x^2 + y^2 + 2cxy$, where none of the quantities a^2, b^2, c^2 is equal to unity, prove that

$$\frac{x^2}{1 - a^2} = \frac{y^2}{1 - b^2} = \frac{z^2}{1 - c^2}.$$

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(ii) Prove that, if x, y, z are any positive numbers,

$$x^2 + y^2 \geqslant 2xy, \quad (y + z)(z + x)(x + y) \geqslant 8xyz.$$

Prove also that, if a, b, c are any three positive numbers such that each is less than the sum of the other two,

$$(b + c - a)(c + a - b)(a + b - c) \leqslant abc.$$

9. If a, b, c, d are any real numbers, prove that

$$a^4 + b^4 \geqslant 2a^2b^2, \quad a^4 + b^4 + c^4 + d^4 \geqslant 4abcd.$$

Prove also that $(a^2 + b^2)^2 + (c^2 + d^2)^2 \geqslant 2(ab + cd)^2$.
Show that, if $a^4 + b^4 + c^4 + d^4 \leqslant 1$, then

$$\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4} + \frac{1}{d^4} \geqslant 16.$$

10. (i) a, b, c, d and x, y, z, u are all real, prove that

$$(a^2 + b^2 + c^2 + d^2)(x^2 + y^2 + z^2 + u^2) \geqslant (ax + by + cz + du)^2.$$

(ii) If $x^3y = aZ^2 + bZ^4$, where $Z \geqslant x$ and a, b, Z, x and y are real and positive, prove that

$$y \geqslant 2(ab)^{\frac{1}{2}}.$$

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11. If a, b, and c are the roots of the equation

$$x^3 + px^2 + qx + r = 0,$$

express the quantities $a + b + c$, $bc + ca + ab$, and abc in terms of p, q, and r.

Prove that (i) $a^2 + b^2 + c^2 = p^2 - 2q$,

(ii) $a^3 + b^3 + c^3 = 3pq - 3r - p^3$,

and, if $abc \neq 0$, (iii) $\dfrac{b}{c} + \dfrac{c}{a} + \dfrac{a}{b} + \dfrac{c}{b} + \dfrac{a}{c} + \dfrac{b}{a} = \dfrac{pq}{r} - 3$.

12. (i) Obtain the equation whose roots exceed by three the roots of the equation $x^4 + 12x^3 + 49x^2 + 78x + 42 = 0$. Hence, or otherwise, solve the original equation.

(ii) If the reciprocals of the roots of the cubic equation

$$x^3 + 3px^2 + 3qx + r = 0$$

are in arithmetical progression, prove that $2q^3 = r(3pq - r)$.

13. A hemispherical hill of radius a, the centre of whose base is O, stands on a horizontal plain. On the slope of the hill, and due south of the summit, there is a vertical flagstaff. From a point P on the plain, and due south of O, the base of the flagstaff is just visible. From a point Q on the plain, due north of O and such that OP = OQ, the top of the flagstaff is just visible.

If OP = OQ = $a + b$, prove that the height of the flagstaff is

$$2a^3(a + b)^{-1}\{b(2a + b)\}^{\frac{1}{2}}.$$

Show that the distance, measured along the hill, from the base of the flagstaff to the summit of the hill is

$$a \sin^{-1}\{a/(a + b)\}.$$

14. Prove that the radius of the circle inscribed in a triangle ABC is equal to $a \sin \frac{1}{2}B \sin \frac{1}{2}C / \cos \frac{1}{2}A$.

A point P is taken in the side BC of the above triangle such that the circles inscribed in the triangles APB and APC are equal; prove that, if θ be the angle between PA and BC, $a \cos \theta = b - c$.

Prove also that the distance between the centres of these circles is $\{\sqrt{s} - \sqrt{(s - a)}\}\sqrt{(s - a)}$.

15. A circle ABC, of radius a, has its centre at O, AOB is a diameter and angle COA $= 2\theta$, where $0 < 2\theta < \pi/2$.

The straight line OC meets the tangent at A in D and the perpendicular from A on OC meets OC in E. Show that

$$\tan \mathrm{EBC} = \tan \mathrm{DBC} = \tan^3 \theta.$$

If $\angle$ EBC $= \angle$DBC $= \alpha$, prove that

$$\frac{\mathrm{EC}}{\mathrm{CD}} = \frac{\sin(\theta - \alpha)}{\sin(\theta + \alpha)},$$

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and prove also that the radius of the circumcircle of the triangle BED is $\frac{1}{2}a(5 + 3\cos 4\theta)\operatorname{cosec} 4\theta$.

16. Prove that, if a convex quadrilateral has sides a, b, c, d and area Δ, then

$$\Delta^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\phi,$$

where $2s = a + b + c + d$ and 2ϕ is the sum of a pair of opposite angles of the quadrilateral.

Deduce the area of the quadrilateral if it circumscribes a circle and show that, if the sides be given, this circle is greatest when the quadrilateral is cyclic.

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CHAPTER II

Series, Exponential and Hyperbolic Functions, Complex Numbers, etc.

1. $$\operatorname*{Lt}_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\,; \quad \operatorname*{Lt}_{\theta \to 0} \frac{\tan\theta}{\theta} = 1.$$

2. $$\left.\begin{aligned} \frac{1}{1-x} &= (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots \to \infty \\ \frac{1}{1+x} &= (1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots \to \infty \end{aligned}\right\} \text{G.P.}$$

$$\left.\begin{aligned} \frac{1}{(1-x)^2} &= (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots \to \infty \\ \frac{1}{(1+x)^2} &= (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots \to \infty \end{aligned}\right\} \text{Arithmetico-geometrical.}$$

All valid for $|x| < 1$.

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3. Exponentials

$$e^x = \operatorname*{Lt}_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots,$$

for all values of x;

$$e = \operatorname*{Lt}_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 2{\cdot}71828\ldots$$

4. Logarithmic Series

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \to \infty \text{ for } -1 < x \leqslant 1\,;$$

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots \to \infty \text{ for } -1 \leqslant x < 1\,;$$

$$\log_e \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots \to \infty\right) \text{ for } -1 < x < 1\,;$$

$$\log_e \frac{n+1}{n-1} = 2\left(\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \ldots \to \infty\right) \text{ for } n > 1\,;$$

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$$\log_e \frac{m+1}{m} = 2\left\{\frac{1}{2m+1} + \frac{1}{3(2m+1)^3} + \frac{1}{5(2m+1)^5} + \ldots \to \infty\right\}$$

for $m > 0$.

5. Hyperbolic Functions

$$\left.\begin{aligned} \sinh x &= \tfrac{1}{2}(e^x - e^{-x}) = x + x^3/3! + x^5/5! + \ldots \to \infty \\ \cosh x &= \tfrac{1}{2}(e^x + e^{-x}) = 1 + x^2/2! + x^4/4! + \ldots \to \infty \end{aligned}\right\},$$

for all values of x;

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x};$$

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}} = \frac{1}{\cosh x};$$

$$\operatorname{cosech} x = \frac{2}{e^x - e^{-x}} = \frac{1}{\sinh x};$$

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}.$$

$$\cosh^2 x - \sinh^2 x = 1;$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x;$$

$$\coth^2 x - 1 = \operatorname{cosech}^2 x.$$

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$$\sinh(-x) = -\sinh x; \quad \cosh(-x) = \cosh x;$$

$$\tanh(-x) = -\tanh x.$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y;$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y;$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}.$$

$$\sinh 2x = 2 \sinh x \cosh x = 2 \tanh x/(1 - \tanh^2 x);$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$$
$$= (1 + \tanh^2 x)/(1 - \tanh^2 x);$$

$$\tanh 2x = 2 \tanh x/(1 + \tanh^2 x).$$

$$\cosh^2 x = \tfrac{1}{2}(\cosh 2x + 1); \quad \sinh^2 x = \tfrac{1}{2}(\cosh 2x - 1).$$

$$\sinh A + \sinh B = 2 \sinh \frac{A+B}{2} \cosh \frac{A-B}{2};$$

$$\sinh A - \sinh B = 2 \cosh \frac{A+B}{2} \sinh \frac{A-B}{2}.$$

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$$\cosh A + \cosh B = 2 \cosh \frac{A+B}{2} \cosh \frac{A-B}{2};$$

$$\cosh A - \cosh B = 2 \sinh \frac{A+B}{2} \sinh \frac{A-B}{2}.$$

$$2 \sinh x \cosh y = \sinh (x+y) + \sinh (x-y);$$

$$2 \cosh x \sinh y = \sinh (x+y) - \sinh (x-y).$$

$$2 \cosh x \cosh y = \cosh (x+y) + \cosh (x-y);$$

$$2 \sinh x \sinh y = \cosh (x+y) - \cosh (x-y).$$

$$\sinh^{-1} x = \log_e \{x + \sqrt{(x^2+1)}\};$$

$$\cosh^{-1} x = \log_e \{x \pm \sqrt{(x^2-1)}\} \quad (x > 1);$$

$$\tanh^{-1} x = \tfrac{1}{2} \log_e \frac{1+x}{1-x} \qquad (x^2 < 1).$$

6. Complex Numbers

$i = \sqrt{(-1)}$ and, if a and b are real $a + ib$ is a complex number whose modulus $r = \sqrt{(a^2+b^2)}$ and whose argument θ is given by

$$\sin \theta = \frac{b}{\sqrt{(a^2+b^2)}}, \quad \cos \theta = \frac{a}{\sqrt{(a^2+b^2)}}.$$

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De Moivre's Theorem: $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$, where n is rational.

$$e^{i\theta} = \cos \theta + i \sin \theta; \quad e^{-i\theta} = \cos \theta - i \sin \theta;$$

$$\cos \theta = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots \to \infty;$$

$$\sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots \to \infty.$$

$$\cos i\theta = \cosh \theta; \quad \sin i\theta = i \sinh \theta; \quad \tan i\theta = i \tanh \theta.$$

$$\cosh i\theta = \cos \theta; \quad \sinh i\theta = i \sin \theta; \quad \tanh i\theta = i \tan \theta.$$

nth roots of $a + ib \equiv r(\cos \theta + i \sin \theta)$, where r is the modulus and θ the argument (amplitude), are given by

$$r^{1/n}\left[\cos \frac{2k\pi + \theta}{n} + i \sin \frac{2k\pi + \theta}{n}\right], \text{ where } k = 0, 1, 2, 3, \ldots, (n-1).$$

$a - ib$ is the conjugate complex number to $a + ib$, and vice versa.

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$$\log_e \frac{m+1}{m} = 2\left\{\frac{1}{2m+1} + \frac{1}{3(2m+1)^3} + \frac{1}{5(2m+1)^5} + \ldots \to \infty\right\}$$

for $m > 0$.

5. Hyperbolic Functions

$$\left.\begin{aligned} \sinh x &= \tfrac{1}{2}(e^x - e^{-x}) = x + x^3/3! + x^5/5! + \ldots \to \infty \\ \cosh x &= \tfrac{1}{2}(e^x + e^{-x}) = 1 + x^2/2! + x^4/4! + \ldots \to \infty \end{aligned}\right\},$$

for all values of x;

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x};$$

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}} = \frac{1}{\cosh x};$$

$$\operatorname{cosech} x = \frac{2}{e^x - e^{-x}} = \frac{1}{\sinh x};$$

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}.$$

$$\cosh^2 x - \sinh^2 x = 1;$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x;$$

$$\coth^2 x - 1 = \operatorname{cosech}^2 x.$$

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$$\sinh(-x) = -\sinh x; \quad \cosh(-x) = \cosh x;$$

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$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y;$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y;$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}.$$

$$\sinh 2x = 2 \sinh x \cosh x = 2 \tanh x/(1 - \tanh^2 x);$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$$
$$= (1 + \tanh^2 x)/(1 - \tanh^2 x);$$

$$\tanh 2x = 2 \tanh x/(1 + \tanh^2 x).$$

$$\cosh^2 x = \tfrac{1}{2}(\cosh 2x + 1); \quad \sinh^2 x = \tfrac{1}{2}(\cosh 2x - 1).$$

$$\sinh A + \sinh B = 2 \sinh \frac{A+B}{2} \cosh \frac{A-B}{2};$$

$$\sinh A - \sinh B = 2 \cosh \frac{A+B}{2} \sinh \frac{A-B}{2}.$$

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EXAMPLES II

1. Write down the series for $\log_e(1+x)$ and state its range of validity. Deduce that, for all positive values of n,

$$\tfrac{1}{2}\log_e\frac{n+1}{n} = \frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots,$$

and hence, or otherwise, show that the ratio of two consecutive terms of the sequence $u_1, u_2, u_3, \ldots, u_n, \ldots$, where $u_n = \dfrac{n!\,e^n}{n^{(2n+1)/2}}$ approaches unity as n increases.

2. (i) Show that the series $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \ldots + (-1)^{n-1}\dfrac{x^n}{n} + \ldots$ converges if $-1 < x < 1$.

By writing $\log_e\left(1+\dfrac{1}{n}\right)$ as $-\log_e\dfrac{n}{n+1}$, or otherwise, prove that, when $n > 1$

$$\log_e\left(1+\frac{1}{n}\right)^n = 1 - \sum_{r=1}^{\infty}\frac{1}{r(r+1)(n+1)^r}.$$

(ii) Prove that, when n is large

$$\left(1+\frac{1}{n}\right)^n = e\left(1 - \frac{1}{2n} + \frac{11}{24n^2} + \ldots\right).$$

3. Write down the nth term of the series for $\log_e(1+x)$ in ascending powers of x and show that the series is convergent when $-1 < x < 1$.

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Prove that, if $p > 1$,

$$\frac{2}{p} + \frac{2}{3p^3} < \log_e\frac{p+1}{p-1} < \frac{2}{p} + \frac{2}{3p^3} + \frac{2}{5p^3(p^2-1)}.$$

By giving p the values 26, 31, 49, calculate to three decimal places the value of $\log_e 5$.

4. Write down the expansions of $\sin x$ and $\cos x$ in ascending powers of x.

A circular arc subtends an angle θ at the centre of the circle and a, b, c are the lengths of the chords of the arc, of two-thirds of the arc and of one-third of the arc respectively. Prove that numbers x, y, z can be found independent of θ, a, b, c such that the length of the arc is $ax + by + cz$ approximately, the approximation being correct to the order θ^6, and determine these values.

5. (*a*) Find in the form $(a+ib)$ all the roots of the equation

$$x^4 + 1 = 0.$$

(*b*) If $Z = z + a^2/z$, prove that, when z describes the circle $x^2 + y^2 = a^2$, Z describes a straight line, and find its length.

Prove that, if z describes the circle $x^2 + y^2 = b^2$, where $b > a$, Z describes an ellipse whose foci are the extremities of the above line.

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6. (i) If $u + iv = (x + iy)^{\frac{1}{2}}$, where u, v are real and the point (x, y) describes the circle $(x - 1)^2 + y^2 = 1$, find the polar equation of the locus of the point (u, v).

(ii) Find the expansion of $e^x \sin x$ in ascending powers of x, giving the first four non-vanishing terms, and prove that the coefficient of x^n is

$$\frac{2^{n/2}}{n!} \sin \frac{n\pi}{4}. \qquad (II)$$

7. (i) The point P represented in the Argand diagram by the complex number $z(= x + iy)$ lies on the straight line whose equation is $6x + 8y = R$, where R is real. Q is the point represented by R^2/z. Show that the locus of Q is a circle and find its centre and radius.

(ii) Express $\sin^7 \theta$ as a sum of sines of multiples of θ. (II)

8. Two complex numbers $x + iy$ and $u + iv$ are connected by the equation

$$\frac{x + iy + c}{x + iy - c} = e^{u+iv},$$

where x, y, u, v are real quantities. Prove that

$$\frac{x}{\sinh u} = \frac{y}{-\sin v} = \frac{c}{\cosh u - \cos v}.$$

9. Prove that, in the Argand diagram, if the centre of a circle of radius r is the point representing the complex number c, the complex number represented by any point on this circle can be expressed by $c + re^{i\theta}$.

In an Argand diagram two opposite vertices of a regular hexagon are respectively the origin and the point representing the complex number $2z$.

Find the general values for the number represented by any point on the circles, (i) circumscribed to the hexagon, (ii) inscribed in the hexagon.

Find also the values of the numbers which are represented by the points at which the inscribed circle touches the sides of the hexagon.

10. Give a geometrical interpretation on the Argand diagram of a rule for multiplying two complex numbers.

If θ be real and n be a positive integer, prove that

$$(\cos \theta + i \sin \theta)^n (\sin \theta + i \cos \theta)^n = e^{in\pi/2}.$$

Express the three values of $(1 + i\sqrt{3})^{\frac{1}{3}}(\sqrt{3} + i)^{\frac{1}{3}}$ in the form $a + ib$, where a and b are real. (The positive value of $\sqrt{3}$ is to be used.)

11. Show how to represent the complex number $z(= x + iy)$ by a point in the Argand diagram.

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(i) If p be real, and the complex number $\dfrac{1+i}{2+ip}+\dfrac{2+3i}{3+i}$ be represented in the Argand diagram by a point on the line $x=y$, show that $p=-5\pm\sqrt{21}$.

(ii) The complex numbers $z_1=x_1+iy_1$, and $z_2=x_2+iy_2$ are connected by the relation $z_1=z_2+\dfrac{1}{z_2}$.

If the point representing z_2 in the Argand diagram describes a circle of radius a and centre at the origin, show that the point representing z_1 describes the ellipse

$$\frac{x^2}{(1+a^2)^2}+\frac{y^2}{(1-a^2)^2}=\frac{1}{a^2}.$$

12. If n be a real, positive integer, prove that one of the values of $(\cos\theta+i\sin\theta)^n$ is $\cos n\theta+i\sin n\theta$.

If $y=e^x\cos^2 x$, where x is real, expand y in a series of ascending powers of x, and prove that the coefficient of x^n is $\dfrac{\frac{1}{2}(1+5^{n/2}\cos n\theta)}{n!}$, where $\theta=\tan^{-1}2$.

13. Explain how the trigonometric and hyperbolic functions may be evaluated for complex values of the variable.

Find the Cartesian equation of the curve in the Argand diagram described by the point $z\equiv x+iy$ when it varies in such a way that $\sinh z-z$ is real.

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Sketch roughly the shape of that part of the curve between the lines $y=\pm\pi$ and show that it asymptotically approaches these lines.

14. Indicate on the Argand diagram the positions of the points $z=1+\sin\theta\pm i\cos\theta$ for a given value of the angle θ.

Prove that one of the values of $\left(\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n$ is equal to $\cos n\left(\dfrac{\pi}{2}-\theta\right)+i\sin n\left(\dfrac{\pi}{2}-\theta\right)$.

Obtain all the values of $\left(\dfrac{\sqrt{2}+1+i}{\sqrt{2}+1-i}\right)^{\frac{1}{2}}$ in the form $a+ib$, where a and b are real.

15. Write down the modulus and amplitude (argument) of $w=e^z$, where $z=x+iy$, and x, y are real numbers.

The point representing z in the Argand diagram describes the triangle with vertices $z=0$, 1, $1+i$. Show that the point representing $w=e^z$ describes a curvilinear triangle which is equiangular with the triangle described by z.

16. Prove that, if $\cos\theta+i\sin\theta=t$, then $2\cos n\theta=t^n+t^{-n}$, $2i\sin n\theta=t^n-t^{-n}$, where n is any integer. Hence, or otherwise, establish the formulae

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$$16 \sin^5 \theta = \sin 5\theta - 5 \sin 3\theta + 10 \sin \theta,$$

$$32 \cos^6 \theta = \cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10.$$

Solve completely the equation

$$\cos 5\theta + 5 \cos 3\theta + 10 \cos \theta = \tfrac{1}{2},$$

where θ is real.

17. Prove that $\tanh x = 1 - \frac{1}{3}x^2 + \frac{2}{15}x^4 \ldots$, and sketch the graph of the function $y = \tanh x$, where x is real. Show that the equation $\tanh x = \lambda x$ has two real non-zero roots if λ is a real number such that $0 < \lambda < 1$.

Define $\tan z$, where z is a complex number, and prove that the equation $\tan z = \lambda z$ has two purely imaginary roots if λ is a real number such that $0 < \lambda < 1$, but that, if λ has any other real value, all the roots of the equation are real.

18. (i) Define $\sin z$ and $\cos z$, where z is complex, and show that $\sin iz = i \sinh z$, $\cos iz = \cosh z$.

Show that the general value of $\cos^{-1} 2$ is $2n\pi \pm i \log_e (2 + \sqrt{3})$, where n is any integer.

(ii) Prove that, if $\tan z = \cos \alpha + i \sin \alpha$, where α is real and acute, then $z = (n + \frac{1}{4})\pi + \frac{1}{2}i \log \left\{\tan \left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right\}$, where n is any integer.

19. Find u and v, the real and imaginary parts of

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$$u + iv = (z - 1)e^{-i\alpha} + e^{i\alpha}/(z - 1),$$

where $z = x + iy$ and α is real.

Prove that the locus of the points on the Argand diagram representing the complex number z such that $v = 0$ is a circle of unit radius with centre at the point (1, 0) and a straight line through the centre of the circle.

20. If the complex numbers z_1 and z_2 are represented in the Argand diagram by the points P and Q respectively, interpret geometrically the modulus and amplitude (argument) of $z_2 - z_1$.

If a third complex number z_3 is represented by the point R, and the angles of the triangle PQR at Q and R are each $\frac{1}{2}(\pi - \alpha)$, prove that $(z_3 - z_2)^2 = 4(z_3 - z_1)(z_1 - z_2) \sin^2 \frac{1}{2}\alpha$.

21. Show how to represent the complex numbers $z(= x + iy)$ and $1/z$ by points in a plane.

The complex numbers z_1 and z_2 are represented by the points P_1 and P_2 respectively. If $z_1(1 - z_2) = z_2$ and P_1 describes the line $2x + 1 = 0$, prove that P_2 describes a circle whose centre is at the origin.

Find the radius of this circle and the sense in which it is described as P_1 moves along the line in the direction which makes its ordinate increase.

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22. If $x + iy = c \tanh (u + iv)$, where x, y, c, u, and v are all real, determine x and y in terms of real functions of u, v, and c.

Prove that this relationship implies both

$$x^2 + y^2 + c^2 - 2cx \coth 2u = 0,$$

and

$$x^2 + y^2 - c^2 + 2cy \cot 2v = 0.$$

23. (i) Show that all the points in the Argand diagram which represent the values of $\log (1 + i)$, lie on a straight line parallel to the imaginary axis. What is the distance between consecutive points?

(ii) In the Argand diagram PQR is an equilateral triangle of which the circumcentre is at the origin. If P represents the number $2 + i$, find the numbers represented by Q and R.

24. Show that every root of the equation $(z + 1)^{2n} + (z - 1)^{2n} = 0$, where n is a positive integer, is purely imaginary.

If the roots are represented in the Argand diagram by points $P_1, P_2, \ldots, P_{2n}$, prove that, if O be the origin,

$$OP_1^2 + OP_2^2 + \ldots + OP_{2n}^2 = 2n(2n - 1).$$

25. Prove De Moivre's Theorem for a positive rational index.

By finding the real quadratic factors of

$$x^{2n} + x^{2n-1} + \ldots + x + 1,$$

where n is a positive integer, show that

$$2^n \sin\frac{\pi}{2n+1} \sin\frac{2\pi}{2n+1} \ldots \sin\frac{n\pi}{2n+1} = \sqrt{(2n+1)}.$$

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26. Solve the equation $(z + 1)^8 - z^8 = 0$, and prove that

$$(z + 1)^8 - z^8 = \tfrac{1}{16}(2z + 1) \prod_{S=1}^{3} \left\{4z^2 + 4z + \operatorname{cosec}^2\left(\frac{S\pi}{8}\right)\right\}.$$

Hence show that

$$16 (\cos^{16}\theta - \sin^{16}\theta) = \cos 2\theta \prod_{S=1}^{3} \left\{\cos^2 2\theta + \cot^2\left(\frac{S\pi}{8}\right)\right\}.$$

27. If n be a positive integer, prove that

$$x^n + \frac{1}{x^n} - 2\cos n\theta = \prod_{r=0}^{n-1} \left\{x + \frac{1}{x} - 2\cos\left(\theta + \frac{2r\pi}{n}\right)\right\}$$

and deduce that $\sin n\theta = 2^{n-1} \prod_{r=0}^{n-1} \sin\left(\theta + \frac{r\pi}{n}\right)$.

Prove also that, if k be a positive integer

$$2^{k-1} \sin\theta \sin\left(\theta + \frac{\pi}{k}\right) \sin\left(\theta + \frac{2\pi}{k}\right) \ldots \sin\left(\theta + \frac{2k-1}{k}\pi\right)$$
$$= (-1)^k (1 - \cos 2k\theta).$$

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28. Prove that, if a and b are any given complex numbers, the locus of a point z in the Argand diagram such that $\left|\dfrac{z-a}{z-b}\right|$ is constant is in general a circle.

Show that the roots of the equation $(z-1)^5 = 32(z+1)^5$ are represented by points lying on a circle of radius $4/3$, and that the values of z are

$$\left(-3+4i\sin\frac{2r\pi}{5}\right)\Big/\left(5-4\cos\frac{2r\pi}{5}\right)\ (r=0,1,\ldots,4).$$

Deduce that

$$\prod_{r=1}^{5}\left(-3+4i\sin\frac{2r\pi}{5}\right) = -\frac{33}{31}\prod_{r=1}^{5}\left(5-4\cos\frac{2r\pi}{5}\right).$$

29. Solve the equation $x^{2n} - 2x^n\cos n\theta + 1 = 0$ and express the left hand side of the equation as a product of real quadratic factors.

Prove that

$$\cos n\phi - \cos n\theta = 2^{n-1}\prod_{r=0}^{n-1}\left\{\cos\phi - \cos\left(\theta+\frac{2r\pi}{n}\right)\right\},$$

and that $$\sin n\theta = 2^{n-1}\prod_{r=0}^{n-1}\sin\left(\theta+\frac{r\pi}{n}\right).$$

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CHAPTER III

Partial Fractions and Summation of Series

Partial Fractions

1. For resolving $f(x)/\phi(x)$ in partial fractions, where $f(x)$ and $\phi(x)$ are polynomials in x and $f(x)$ is of lower degree than $\phi(x)$ in x:—

To every linear factor $(a_1x + b_1)$ of $\phi(x)$ there will be a partial fraction $A_1/(a_1x + b_1)$.

To every quadratic factor $(a_2x^2 + b_2x + c_2)$ of $\phi(x)$ there will be a partial fraction $$\frac{A_2x + B_2}{a_2x^2 + b_2x + c_2}.$$

To every factor of the form $(a_3x + b_3)^2$ of $\phi(x)$ there will be two partial fractions $$\frac{A_3}{a_3x + b_3} + \frac{B_3}{(a_3x + b_3)^2}.$$

To every factor of the form $(a_4x^2 + b_4x + c_4)^2$ of $\phi(x)$ there will be two partial fractions

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$$\frac{A_4x + B_4}{a_4x^2 + b_4x + c_4} + \frac{C_4x + D_4}{(a_4x^2 + b_4x + c_4)^2}; \text{ and so on.}$$

If $f(x)$ is of the same degree as $\phi(x)$ in x, then A must be added to the above partial fractions.

If $f(x)$ is one degree higher than $\phi(x)$ in x, then $Ax + B$ must be added to the above partial fractions; and so on.

All the constants A, B, A_1, B_1, A_2, B_2, etc., are determined by using the properties of identities.

2. The results for the summation of series can be obtained by making full use of the series stated and developed in Chapters I and II, together with the method of induction.

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EXAMPLES III

1. Resolve into partial fractions $x^2/(1 + x^2)(1 - x)^2$.

   Find the sum of the coefficients up to and including that of x^{2n} in the expansion of the above function in ascending powers of x when $|x| < 1$.

2. Express $(a-b)^2/(1-ax)^2(1-bx)$ in terms of partial fractions.

If the function be expanded in positive integral powers of x, find the coefficient of x^n and state the range of values of x for which the expansion is valid.

Deduce, or prove by any method, that the expression

$$(n+1)a^{n+2}-(n+2)a^{n+1}b+b^{n+2},$$

where n is a positive integer contains $(a-b)^2$ as a factor.

3. Express in partial fractions the function $\dfrac{1-t^2}{(1-at)(1-bt)}$, where the non-zero constants a, b are (i) unequal, (ii) equal.

By taking $a=1/b=e^{i\theta}$, and a suitable value of t deduce, or otherwise prove that, if $0<\phi<\frac{1}{2}\pi$,

$$\frac{\cos\phi}{1-\sin\phi\cos\theta}=1+\sum_{n=1}^{\infty}2\tan^n\tfrac{1}{2}\phi\cos n\theta.$$

4. Prove that the binomial series

$$1+mx+\frac{m(m-1)}{2!}x^2+\frac{m(m-1)(m-2)}{3!}x^3+\ldots$$

converges if $-1<x<+1$.

Sum the series $\frac{1}{2}t+\dfrac{1}{2\,.\,4}t^3+\dfrac{1\,.\,3}{2\,.\,4\,.\,6}t^5+\ldots$ when convergent.

If $t=2x/(1+x^2)$, show that for a given value of t there are two values of x which are reciprocals, and hence prove that the series

$$\frac{1}{2}\left(\frac{2x}{1+x^2}\right)+\frac{1}{2\,.\,4}\left(\frac{2x}{1+x^2}\right)^3+\frac{1\,.\,3}{2\,.\,4\,.\,6}\left(\frac{2x}{1+x^2}\right)^5+\ldots$$

converges to x or $1/x$ according as $|x|$ is less or greater than unity.

5. (i) Given that $0<n<1$, prove that the infinite series

$$1+nx+\frac{n(n-1)}{2!}x^2+\ldots+\frac{n(n-1)\ldots(n-r+1)}{r!}x^r+\ldots$$

is convergent if $|x|<1$.

(ii) Calculate without the use of tables, the value of $65^{\frac{1}{3}}$ to four places of decimals.

(iii) Find the sum to infinity of the series whose nth term is

$$(-1)^{n-1}\frac{2\,.\,5\,.\,8\ldots(3n-1)}{6\,.\,9\,.\,12\ldots(3n+3)}\cdot\frac{1}{3^{2n+3}}.$$

6. (*a*) Prove that the sums of the infinite series whose nth terms are

$$\text{(i)}\ \frac{1}{n(n+1)}\cdot\frac{1}{2^n},\qquad \text{(ii)}\ \frac{n^2-2n+3}{n!}$$

are respectively $1-\log_e 2$ and $3(e-1)$.

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(*b*) Find the sum of the infinite series whose nth term is $\sin n\theta \cos^n \theta$, where $0 < \theta < \pi/2$.

7. Find the sum to n terms of the series whose rth term is

$$\frac{1}{(3r-2)(3r+1)(3r+4)},$$

and find the sum to infinity.

Find also the sum to infinity of the series whose rth terms are

$$\text{(i)}\ \frac{(r+1)^2}{r!}, \qquad \text{(ii)}\ \frac{\cos r\theta}{2^r + 2^{r+1}}.$$

8. If $u_n = \dfrac{n^2+9n+5}{(n+1)(2n+3)(2n+5)(n+4)}$, prove that $\sum_{n=1}^{\infty} u_n = 5/36$.

9. Prove that the infinite series whose nth term is $\dfrac{x^n}{n(n+1)(n+2)}$ converges when $-1 \leqslant x \leqslant 1$, and that its sum when $-1 \leqslant x < 1$ is $\frac{3}{4} - \dfrac{1}{2x} - \dfrac{(1-x)^2}{2x^2}\log_e(1-x)$. Find the sum when $x = 1$.

10. Write down the expansions for $\log_e \dfrac{1+x}{1-x}$ and $\tan^{-1} x$ in ascending powers of x, stating the values of x for which the expansions are valid.

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If $0 < x < 1$ and $0 < y < 1$ and $\log_e y + 2\sum_{n=1}^{\infty} \dfrac{x^{2n-1}}{2n-1} = 0$, express y in terms of x, and deduce that,

$$\log_e x + 2\sum_{n=1}^{\infty} \frac{y^{2n-1}}{2n-1} = 0.$$

Prove also that $\sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^{2n-1} + y^{2n-1}}{2n-1} = \dfrac{\pi}{4}$.

11. (*a*) Show how to evaluate the sum $\sum_{r=1}^{n} u_r$, given that $u_r = v_r - v_{r-1}$, where v_r is a known function of r.

Hence sum to n terms the series whose rth term is

$$\text{(i)}\ \frac{1}{r(r+1)(r+2)(r+3)}, \qquad \text{(ii)}\ \operatorname{cosec} 2^r\theta.$$

(*b*) Sum to infinity the series whose rth term is $\dfrac{\sin^2 r\theta}{r!}$.

12. Evaluate (i) $\sum_{p=1}^{n} \dfrac{p-2}{p(p+1)(p+3)}$, (ii) $\sum_{p=1}^{n} p \sin p\theta$.

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13. Sum to n terms the series of which the rth term is $(2r-1)x^{r-1}$. If $\alpha = 2\pi/n$, where n is a positive integer, show that

$$1 + 3\cos\alpha + 5\cos 2\alpha + \ldots + (2n-1)\cos(n-1)\alpha = -n,$$

$$3\sin\alpha + 5\sin 2\alpha + \ldots + (2n-1)\sin(n-1)\alpha = -n\cot\tfrac{1}{2}\alpha.$$

14. (i) Express $\sin 3x$ in terms of $\sin x$ and find the sum to n terms and infinity of the series whose rth term is $3^{r-1}\sin^3(\theta/3^r)$.

(ii) If $b \leqslant a$, sum to infinity the series

$$\frac{b}{a}\sin C + \frac{b^3}{2a^3}\sin 2C + \frac{b^3}{3a^3}\sin 3C + \ldots$$

If a, b, c, A, B, C are the elements of a plane triangle, prove that the sum is equal to $B + n\pi$, where n is any integer.

15. Sketch the graph of the curve $y = \cot^{-1} x$, for values of y lying between $-\pi$ and $+\pi$, the inverse function being regarded as many valued.

If $a > 0$ and the inverse functions are interpreted as acute angles, prove that

$$\tan^{-1}(1+a) - \tan^{-1} a = \cot^{-1}(1 + a + a^2).$$

Hence prove that, if n be a positive integer,

$$\cot^{-1} 3 + \cot^{-1} 7 + \cot^{-1} 13 + \ldots + \cot^{-1}(1 + n + n^2) = \cot^{-1}(1 + 2/n),$$

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and deduce that the infinite series $\sum_{n=1}^{\infty} \cot^{-1} n^2$ converges.

16. Prove that, if n be an odd integer,

$$2^{n-1}\cos^n\theta = \cos n\theta + {}^nC_1\cos(n-2)\theta + {}^nC_2\cos(n-4)\theta + \ldots + {}^nC_{\frac{1}{2}(n-1)}\cos\theta.$$

Prove that $$16\sum_{r=1}^{5}\cos^5\frac{2r\pi}{5} = 5.$$

17. Prove that

$$\sin\alpha - \sin(\alpha+\beta) + \sin(\alpha+2\beta) - \ldots - \sin\{\alpha + (2n-1)\beta\} = -\sin n\beta\cos\{\alpha + (n-\tfrac{1}{2})\beta\}\sec\beta/2,$$

where α and β are real constants.

$A_1, A_2, \ldots, A_{2n}$ are the vertices of a regular polygon of $2n$ sides. Prove that the sum of the lengths $A_1A_3, A_1A_5, \ldots, A_1A_{2n-1}$ differs from the sum of the lengths $A_1A_2, A_1A_4, \ldots, A_1A_{2n}$ by $a\sec^2\pi/4n$, where $2a$ is the length of the side of the polygon.

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CHAPTER IV

Differentiation

1. Definitions and Theorems

If $y = f(x)$, and u, v, w, z are functions of x, whilst a, b, c, n are constants

$$\frac{dy}{dx} = f'(x) = \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{Lt}} \frac{f(x + \delta x) - f(x)}{\delta x};$$

$$\frac{d}{dx}(u + v - w + \ldots) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx} + \ldots;$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx};$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx};$$

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$$\frac{d}{dx}(u/v) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2};$$

$$\frac{d}{dx}\{af(x)\} = a\frac{d}{dx}\{f(x)\};$$

$$\frac{d}{dx}\{f(ax + b)\} = a\frac{d}{dz}\{f(z)\}, \text{ where } z = ax + b;$$

$$\frac{dx}{dy} = 1\Big/\frac{dy}{dx};$$

Slope of tangent to curve $y = f(x)$ at (x, y) is $\frac{dy}{dx}$.

2. Derivatives of algebraic functions

$$\frac{d}{dx}(\text{C}) = 0, \quad \text{C any constant};$$

$$\frac{d}{dx}(x^n) = nx^{n-1}; \quad \frac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}.$$

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3. Derivatives of trigonometric functions (x in radians)

$$\frac{d}{dx}\{\sin(ax+b)\} = a\cos(ax+b)\,;\quad \frac{d}{dx}(\sin x) = \cos x.$$

$$\frac{d}{dx}\{\cos(ax+b)\} = -\,a\sin(ax+b)\,;\quad \frac{d}{dx}(\cos x) = -\sin x.$$

$$\frac{d}{dx}\{\tan(ax+b)\} = a\sec^2(ax+b)\,;\quad \frac{d}{dx}(\tan x) = \sec^2 x.$$

$$\frac{d}{dx}\{\sec(ax+b)\} = a\sec(ax+b)\tan(ax+b)\,;$$

$$\frac{d}{dx}(\sec x) = \sec x\tan x.$$

$$\frac{d}{dx}\{\operatorname{cosec}(ax+b)\} = -\,a\operatorname{cosec}(ax+b)\cot(ax+b)\,;$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x\cot x.$$

$$\frac{d}{dx}\{\cot(ax+b)\} = -\,a\operatorname{cosec}^2(ax+b)\,;\quad \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x.$$

4. Derivatives of exponential and logarithmic functions

$$\frac{d}{dx}(e^{ax+b}) = ae^{ax+b}\,;\quad \frac{d}{dx}(e^x) = e^x.$$

$$\frac{d}{dx}(a^{bx+c}) = ba^{bx+c}\log_e a\,;\quad \frac{d}{dx}(a^x) = a^x\log_e a.$$

$$\frac{d}{dx}\{\log_e(ax+b)\} = \frac{a}{ax+b}\,;\quad \frac{d}{dx}(\log_e x) = 1/x.$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x\log_e a}.$$

5. Derivatives of Hyperbolic functions

$$\frac{d}{dx}\{\sinh(ax+b)\} = a\cosh(ax+b)\,;\quad \frac{d}{dx}(\sinh x) = \cosh x.$$

$$\frac{d}{dx}\{\cosh(ax+b)\} = a\sinh(ax+b)\,;\quad \frac{d}{dx}(\cosh x) = \sinh x.$$

$$\frac{d}{dx}\{\tanh(ax+b)\} = a\operatorname{sech}^2(ax+b)\,;\quad \frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x.$$

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$$\frac{d}{dx}\{\text{sech}\,(ax+b)\} = -\,a\,\text{sech}\,(ax+b)\tanh\,(ax+b)\,;$$

$$\frac{d}{dx}(\text{sech}\,x) = -\,\text{sech}\,x\tanh x.$$

$$\frac{d}{dx}\{\text{cosech}\,(ax+b)\} = -\,a\,\text{cosech}\,(ax+b)\coth\,(ax+b)\,;$$

$$\frac{d}{dx}(\text{cosech}\,x) = -\,\text{cosech}\,x\coth x.$$

$$\frac{d}{dx}\{\coth\,(ax+b)\} = -\,a\,\text{cosech}^2\,(ax+b)\,;$$

$$\frac{d}{dx}(\coth x) = -\,\text{cosech}^2\,x.$$

6. Derivatives of Inverse Trigonometric functions

$$\frac{d}{dx}(\sin^{-1}x/a) = \frac{1}{\sqrt{(a^2-x^2)}}\,;\ \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{(1-x^2)}}.$$

$$\frac{d}{dx}(\cos^{-1}x/a) = \frac{-1}{\sqrt{(a^2-x^2)}}\,;\ \frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{(1-x^2)}}.$$

$$\frac{d}{dx}(\sec^{-1}x/a) = \frac{a}{x\sqrt{(x^2-a^2)}}\,;\ \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{(x^2-1)}}.$$

$$\frac{d}{dx}(\text{cosec}^{-1}x/a) = \frac{-a}{x\sqrt{(x^2-a^2)}}\,;\ \frac{d}{dx}(\text{cosec}^{-1}x) = \frac{-1}{x\sqrt{(x^2-1)}}.$$

$$\frac{d}{dx}(\cot^{-1}x/a) = \frac{-a}{a^2+x^2}\,;\ \frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}.$$

$$\frac{d}{dx}(\tan^{-1}x/a) = \frac{a}{a^2+x^2}\,;\ \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}.$$

7. Derivatives of Inverse Hyperbolic functions

$$\frac{d}{dx}(\sinh^{-1}x/a) = \frac{1}{\sqrt{(x^2+a^2)}}\,;\ \frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{(x^2+1)}}.$$

$$\frac{d}{dx}(\cosh^{-1}x/a) = \frac{\pm 1}{\sqrt{(x^2-a^2)}}\,;\ \frac{d}{dx}(\cosh^{-1}x) = \frac{\pm 1}{\sqrt{(x^2-1)}}.$$

$$\frac{d}{dx}(\text{sech}^{-1}x/a) = \frac{\pm a}{x\sqrt{(a^2-x^2)}}\,;\ \frac{d}{dx}(\text{sech}^{-1}x) = \frac{\pm 1}{x\sqrt{(1-x^2)}}.$$

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$$\frac{d}{dx}(\operatorname{cosech}^{-1} x/a) = \frac{\pm a}{x\sqrt{(a^2 + x^2)}}; \quad \frac{d}{dx}(\operatorname{cosech}^{-1} x) = \frac{\pm 1}{x\sqrt{(1 + x^2)}}.$$

$$\frac{d}{dx}(\coth^{-1} x/a) = \frac{-a}{x^2 - a^2}; \quad \frac{d}{dx}(\coth^{-1} x) = \frac{-1}{x^2 - 1}.$$
$$(x^2 > a^2) \qquad\qquad (x^2 > 1)$$

$$\frac{d}{dx}(\tanh^{-1} x/a) = \frac{a}{a^2 - x^2}; \quad \frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1 - x^2}.$$
$$(x^2 < a^2) \qquad\qquad (x^2 < 1)$$

8. Successive differentiation

$$\frac{d^n}{dx^n}(e^{ax+b}) = a^n e^{ax+b}$$

$$\frac{d^n}{dx^n}(a^{bx+c}) = (b \log_e a)^n a^{bx+c}.$$

$$\frac{d^n}{dx^n}\left(\frac{1}{ax + b}\right) = \frac{(-1)^n n!\, a^n}{(ax + b)^{n+1}}.$$

$$\frac{d^n}{dx^n}\{\log_e (ax + b)\} = \frac{(-1)^{n-1}(n - 1)!\, a^n}{(ax + b)^n}.$$

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$$\frac{d^n}{dx^n}\{\sin (ax + b)\} = a^n \sin\left(ax + b + \frac{n\pi}{2}\right).$$

$$\frac{d^n}{dx^n}\{\cos (ax + b)\} = a^n \cos\left(ax + b + \frac{n\pi}{2}\right).$$

If $r = \sqrt{(a^2 + b^2)}$, $\cos \alpha = a/\sqrt{(a^2 + b^2)}$, $\sin \alpha = b/\sqrt{(a^2 + b^2)}$,

then
$$\frac{d^n}{dx^n}(e^{ax} \sin \overline{bx + c}) = r^n e^{ax} \sin (bx + c + n\alpha),$$

$$\frac{d^n}{dx^n}(e^{ax} \cos \overline{bx + c}) = r^n e^{ax} \cos (bx + c + n\alpha).$$

Leibnitz's Theorem, where u and v are functions of x, states that

$$\frac{d^n}{dx^n}(uv) = uv_n + {}^nC_1 u_1 v_{n-1} + {}^nC_2 u_2 v_{n-2} + \ldots$$
$$+ {}^nC_r u_r v_{n-r} + \ldots + u_n v,$$

where
$$u_r = \frac{d^r}{dx^r}(u), \text{ etc.}$$

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Derivatives for Parametric Equations

9. If $x = f(t)$, $y = \phi(t)$, $\dot{x} = \dfrac{dx}{dt}$, $\dot{y} = \dfrac{dy}{dt}$, $\ddot{x} = d^2x/dt^2$, $\ddot{y} = d^2y/dt^2$,

then
$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}, \quad \frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x})^3}.$$

EXAMPLES IV

1. Find the following limits :

(i) $\underset{x\to 0}{\text{Lt}} \dfrac{x^2}{1 - \cos x}$, (ii) $\underset{x\to 1}{\text{Lt}} \left\{\dfrac{x}{x-1} - \dfrac{1}{\log_e x}\right\}$,

(iii) $\underset{x\to \pi/4}{\text{Lt}} \{(\tan x)^{\tan 2x}\}$.

2. (i) Evaluate $\underset{x\to 3}{\text{Lt}} \dfrac{x^2 - 9}{\sqrt{(2x+3)} - \sqrt{(4x-3)}}$.

(ii) Find, in its simplest form, the differential coefficient with respect to x of $\left(\dfrac{x}{x + \sqrt{\{a^2 - x^2\}}}\right)^n$.

(iii) If $x = \cos\theta(1 + \cos\theta)$, $y = \sin\theta(1 + \cos\theta)$, find $\dfrac{d^2y}{dx^2}$ in terms of θ. *(I)*

3. (i) Evaluate (a) $\underset{x\to 0}{\text{Lt}} \dfrac{e^x - 1}{\log_e(1+x)}$, (b) $\underset{x\to 0}{\text{Lt}} \dfrac{x^2 \tan x}{\sqrt{(1-x^2)} - 1}$.

(ii) If $y = x \log_e y$, prove that
$$\frac{d^2y}{dx^2} = \frac{y^2(y - 2x)}{x(y-x)^3}.$$ *(I)*

4. Differentiate with respect to x

(i) $x^n e^{\cos x}$, (ii) $\log_e \dfrac{x}{(1+x)^{\frac{1}{2}}(1+x^2)^{\frac{1}{4}}} - \frac{1}{2}\tan^{-1} x$,

and express each result in its simplest form.

If $y = A \sinh x + B \tanh x$, determine A and B so that
$$\frac{d^2y}{dx^2} = (1 - \cosh^3 x)\tanh x \operatorname{sech}^2 x.$$

5. Differentiate with respect to x
$$\sqrt{\left(\frac{a^2 + ax + x^2}{a^2 - ax + x^2}\right)}, \quad (\cosh x)^x,$$
expressing each result in its simplest form.

If $y = \tan^{-1}(\sinh x)$ prove that
$$\frac{d^2y}{dx^2} + \tan y\left(\frac{dy}{dx}\right)^2 = 0.$$

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6. State and prove the rule for differentiating a product of two functions of a single variable.

Differentiate with respect to x

$$\log_e (1 + \sin 2x) + 2 \log_e \left\{\sec \left(\frac{\pi}{4} - x\right)\right\},$$

and express the result in its simplest form. Explain why the result is of such simple form.

7. Differentiate with respect to x the following functions :

(i) $\dfrac{1 - x}{(1 - x^2)^{\frac{1}{2}}}$; (ii) $\tan^{-1} (m \tan x)$; (iii) $\sec \{\frac{1}{2} \log (a^2 + x^2)\}$.

8. Define the derivative of a function and from the definition find the derivative of $\sin x^2$.

Find (i) $\dfrac{dy}{dx}$ if $y = \sin (x + y)^2$.

(ii) $\dfrac{d^2y}{dx^2}$ if $x = 3 \cos \theta - \cos 3\theta$, $y = 3 \sin \theta - \sin 3\theta$.

9. (*a*) Define the derivative of a function and from your definition, assuming the exponential theorem, prove that the derivative of a^x is $a^x \log_e a$.

(*b*) If $y = t^m + t^{-m}$ and $x = t + t^{-1}$, prove that

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(i) $(x^2 - 4)\left(\dfrac{dy}{dx}\right)^2 = m^2(y^2 - 4)$,

(ii) $(x^2 - 4)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - m^2y = 0$.

10. Differentiate with respect to x

(i) $\cos^{-1} (2x \sqrt{1 - x^2})$, (ii) $\tan^{-1} \left(\dfrac{\cos x - \sin x}{\cos x + \sin x}\right)$

reducing the result in each case to its simplest form.

If $y = ax \sin b/x$, prove that $x^4 \dfrac{d^2y}{dx^2} + b^2y = 0$.

11. (*a*) Show from first principles that $\dfrac{d}{d\theta}(\sec \theta) = \sec \theta \tan \theta$.

(*b*) If $y = A \sin \alpha x$, show that

$$\frac{d^ny}{dx^n} = \alpha^n A \sin \left(\alpha x + n\frac{\pi}{2}\right).$$

(*c*) If $z = x^2 \sin \alpha x$, find $\dfrac{d^nz}{dx^n}$. (*I*)

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12. If $y = (1 + x^2)^{\frac{1}{2}} \log \{x + (1 + x^2)^{\frac{1}{2}}\}$, prove that

$$(1 + x^2)(dy/dx - 1) = xy.$$

Hence, or otherwise, assuming that the expression for y can be expanded as a series in ascending powers of x when $|x| < 1$, prove that this series is

$$y = x + \frac{x^3}{3} - \frac{2}{3} \cdot \frac{x^5}{5} + \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{x^7}{7} - \ldots$$

and find the term in x^{2n+1}.

13. If $y = (\sin^{-1} x)^2 + a \sin^{-1} x$, where a is a constant, prove that

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0.$$

Find the expansion of y in positive integral powers of x and give the coefficients of the terms in x^{2p} and x^{2p+1}.

14. If $y = \sin (m \sin^{-1} x)$, where m is a real constant > 1, prove that

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0.$$

If $1 > x > -1$, prove that

$$y = m\left\{x + \frac{1^2 - m^2}{3!}x^3 + \frac{(3^2 - m^2)(1^2 - m^2)}{5!}x^5 + \ldots\right\}$$

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and find the term in x^{2n+1}.

15. If $y = \sin n\theta \operatorname{cosec} \theta$ and $x = \cos \theta$, prove that

$$(1 - x^2)\frac{dy}{dx} - xy + n \cos n\theta = 0,$$

and

$$(1 - x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + (n^2 - 1)y = 0.$$

Show that, if $n = 7$, the latter equation is satisfied by a polynomial of the form $x^6 + bx^4 + cx^2 + d$, and find the values of b, c, and d.

16. If $y = e^{x+3x^2}$, prove that $\dfrac{dy}{dx} = (1 + 6x)y$.

Assuming this relation and also that

$$e^{x+3x^2} = a_0 + a_1x + \frac{a_2}{2!}x^2 + \frac{a_3}{3!}x^3 + \ldots$$

prove that $a_{n+1} - a_n - 6na_{n-1} = 0$.

Hence, or otherwise, obtain the expansion of y as far as the term in x^5.

(I)

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17. If $y = e^{ax} \cos bx$, prove that $d^n y/dx^n = r^n e^{ax} \cos (bx + n\alpha)$, where $r^2 = a^2 + b^2$ and $a \tan \alpha = b$.

Find the value of $d^n y/dx^n$ by employing Leibnitz's theorem, and hence prove that

$$r^n \cos (bx + n\alpha) = a^n \cos bx + na^{n-1}b \cos \left(bx + \frac{\pi}{2}\right)$$

$$+ \frac{n(n-1)}{2!} a^{n-2}b^2 \cos (bx + \pi) + \ldots + b^n \cos \left(bx + \frac{n\pi}{2}\right). \quad (I)$$

18. If $y = e^{x+x^2}$, prove that

$$\frac{d^{n+1}y}{dx^{n+1}} = (2x + 1)\frac{d^n y}{dx^n} + 2n\frac{d^{n-1}y}{dx^{n-1}}.$$

If the expansion of y be written in the form $\sum_{n=0}^{\infty} a_n x^n$, prove that $(n + 1)a_{n+1} = a_n + 2a_{n-1}$ $(n \geqslant 1)$.

Show also that

$$\text{(i)} \sum_{n=0}^{\infty} a_n = e^2, \quad \text{(ii)} \sum_{n=1}^{\infty} na_n = 3e^2, \quad \text{(iii)} \sum_{n=1}^{\infty} n^2 a_n = 14e^2.$$

19. If $y = \cos \{\log (1 + x)\}$, prove that

$$(1 + x)^2 y_{n+2} + (2n + 1)(1 + x)y_{n+1} + (n^2 + 1)y_n = 0,$$

where y_r denotes $d^r y/dx^r$.

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Show that, if $\cos \{\log (1 + x)\}$ can be represented by the power series $a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$, then

$$(n + 1)(n + 2)a_{n+2} + (2n + 1)(n + 1)a_{n+1} + (n^2 + 1)a_n = 0,$$

and determine the expansion up to and including the term in x^6.

20. State and prove Leibnitz's theorem for the nth derivative of the product of two functions.

Prove that $D^{n+1}(xy) = (n + 1)D^n y + xD^{n+1}y$, where $D \equiv d/dx$. By taking $y = x^{n-1}e^{1/x}$, prove, by induction, that

$$D^n(x^{n-1}e^{1/x}) = (-1)^n \frac{e^{1/x}}{x^{n+1}}.$$

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CHAPTER V

Integration

1. Algebraic, Exponential, and Logarithmic functions

$$\int x^n dx = x^{n+1}/(n+1)\ (n \neq -1)\ ; \quad \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{n+1},$$

$$(n \neq -1).$$

$$\int \frac{1}{x} dx = \log_e x\ ; \qquad \int \frac{dx}{ax+b} = \frac{1}{a} \log_e (ax+b).$$

$$\int e^x dx = e^x\ ; \qquad \int e^{ax+b} dx = \frac{1}{a} e^{ax+b}.$$

$$\int a^x dx = \frac{a^x}{\log_e a}\ ; \qquad \int a^{bx+c} dx = \frac{1}{b \log_e a} \cdot a^{bx+c}.$$

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2. Trigonometric functions

$$\int \cos x dx = \sin x\ ; \quad \int \cos (ax+b) dx = \frac{1}{a} \sin (ax+b).$$

$$\int \sin x dx = -\cos x\ ; \quad \int \sin (ax+b) dx = \frac{-1}{a} \cos (ax+b).$$

$$\int \sec^2 x dx = \tan x\ ; \quad \int \sec^2 (ax+b) dx = \frac{1}{a} \tan (ax+b).$$

$$\int \sec x \tan x dx = \sec x\ ; \quad \int \sec (ax+b) \tan (ax+b) dx = \frac{1}{a} \sec (ax+b).$$

$$\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x\ ; \quad \int \operatorname{cosec} (ax+b) \cot (ax+b) dx$$

$$= \frac{-1}{a} \operatorname{cosec} (ax+b).$$

$$\int \operatorname{cosec}^2 x dx = -\cot x\ ; \quad \int \operatorname{cosec}^2 (ax+b) dx = \frac{-1}{a} \cot (ax+b).$$

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3. Hyperbolic functions

$$\int \cosh x dx = \sinh x\,; \quad \int \cosh (ax + b) dx = \frac{1}{a} \sinh (ax + b).$$

$$\int \sinh x dx = \cosh x\,; \quad \int \sinh (ax + b) dx = \frac{1}{a} \cosh (ax + b).$$

$$\int \operatorname{sech}^2 x dx = \tanh x\,; \quad \int \operatorname{sech}^2 (ax + b) dx = \frac{1}{a} \tanh (ax + b).$$

$$\int \operatorname{sech} x \tanh x dx = - \operatorname{sech} x\,;$$

$$\int \operatorname{sech} (ax + b) \tanh (ax + b) dx = \frac{-1}{a} \operatorname{sech} (ax + b).$$

$$\int \operatorname{cosech} x \coth x dx = - \operatorname{cosech} x\,;$$

$$\int \operatorname{cosech} (ax + b) \coth (ax + b) dx = \frac{-1}{a} \operatorname{cosech} (ax + b).$$

$$\int \operatorname{cosech}^2 x dx = - \coth x\,; \quad \int \operatorname{cosech}^2 (ax + b) dx = \frac{-1}{a} \coth (ax + b).$$

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4. Inverse Trigonometric results

$$\int \frac{dx}{\sqrt{(1 - x^2)}} = \sin^{-1} x \text{ or } - \cos^{-1} x\,;$$

$$\int \frac{dx}{\sqrt{(a^2 - x^2)}} = \sin^{-1} x/a \text{ or } - \cos^{-1} x/a.$$

$$\int \frac{dx}{x \sqrt{(x^2 - 1)}} = \sec^{-1} x \text{ or } - \operatorname{cosec}^{-1} x\,;$$

$$\int \frac{dx}{x \sqrt{(x^2 - a^2)}} = \frac{1}{a} \sec^{-1} x/a \text{ or } \frac{-1}{a} \operatorname{cosec}^{-1} x/a.$$

$$\int \frac{dx}{1 + x^2} = \tan^{-1} x \text{ or } - \cot^{-1} x\,;$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} x/a \text{ or } \frac{-1}{a} \cot^{-1} x/a.$$

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5. Inverse Hyperbolic results

$$\int \frac{dx}{\sqrt{(x^2+1)}} = \sinh^{-1} x\,; \quad \int \frac{dx}{\sqrt{(x^2+a^2)}} = \sinh^{-1} x/a.$$

$$\int \frac{dx}{\sqrt{(x^2-1)}} = \pm \cosh^{-1} x\,; \int \frac{dx}{\sqrt{(x^2-a^2)}} = \pm \cosh^{-1} x/a.$$

$$\int \frac{dx}{1-x^2} = \tanh^{-1} x = \frac{1}{2} \log_e \frac{1+x}{1-x} \quad (x^2 < 1)\,;$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{a} \tanh^{-1} x/a = \frac{1}{2a} \log_e \frac{a+x}{a-x} \, (x^2 < a^2).$$

$$\int \frac{dx}{x\sqrt{(1-x^2)}} = \pm \operatorname{sech}^{-1} x\,; \quad \int \frac{dx}{x\sqrt{(a^2-x^2)}} = \pm \frac{1}{a} \operatorname{sech}^{-1} x/a.$$

$$\int \frac{dx}{x\sqrt{(1+x^2)}} = \pm \operatorname{cosech}^{-1} x\,; \quad \int \frac{dx}{x\sqrt{(a^2+x^2)}} = \pm \frac{1}{a} \operatorname{cosech}^{-1} x/a.$$

$$\int \frac{dx}{x^2-1} = -\coth^{-1} x = \frac{1}{a} \log_e \frac{x-1}{x+1} \, (x^2 > 1)\,;$$

$$\int \frac{dx}{x^2-a^2} = -\frac{1}{a} \coth^{-1} x/a = \frac{1}{2a} \log_e \frac{x-a}{x+a} \, (x^2 > a^2).$$

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6. $$\int \frac{f'(x)}{f(x)} dx = \log_e \{f(x)\}.$$

$$\int \tan x \, dx = \log_e (\sec x)\,; \quad \int \cot x \, dx = \log_e (\sin x).$$

$$\int \tanh x \, dx = \log_e (\cosh x)\,; \quad \int \coth x \, dx = \log_e (\sinh x).$$

$$\int \operatorname{cosec} x \, dx = \log_e (\tan \tfrac{1}{2}x)\,; \quad \int \sec x \, dx = \log_e \{\tan (\tfrac{1}{4}\pi + \tfrac{1}{2}x)\}.$$

7. $$\int \frac{f'(x)}{\sqrt{\{f(x)\}}} dx = 2\sqrt{\{f(x)\}}.$$

$$\int \frac{x\,dx}{\sqrt{(x^2 \pm 1)}} = \sqrt{(x^2 \pm 1)}\,; \quad \int \frac{x\,dx}{\sqrt{(1-x^2)}} = -\sqrt{(1-x^2)}.$$

Note.—All the previous integrals are taken as indefinite integrals and if the general integral is required a constant of integration C will be added.

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8. Suggested substitutions for integration by substitution

Type of integrand	*Suggested substitution*
Expression containing $\sqrt{(a^2 - x^2)}$	$x = a \sin\theta$, or $x = a \tanh\theta$.
Expression containing $\sqrt{(a^2 + x^2)}$	$x = a \sinh\theta$, $x = a \tan\theta$.
Expression containing $\sqrt{(x^2 - a^2)}$	$x = a \cosh\theta$, $x = a \sec\theta$.
Expression containing fractional powers of x	$x = z^r$, where r is the L.C.M. of the denominators of the fractional indices.
$xf(x^2)$	$x^2 = z$.
Function of x and $\sqrt{(px + q)}$	$\sqrt{(px + q)} = z$, i.e. $px + q = z^2$.
Function of x and $\sqrt{(x - a)(x - b)}$	$\sqrt{(x - b)} = z\sqrt{(x - a)}$.
Function of x and $\sqrt{(b - x)(x - a)}$	$\sqrt{(b - x)} = z\sqrt{(x - a)}$.
Function of x and $\sqrt{(ax^2 + bx + c)}$	$x + \sqrt{(x^2 + bx + c)} = z$.

9. Integration by parts formula

$$\int uv\,dx = u\int v dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx.$$

10. Reduction formula

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If $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\,dx$, and m, n are positive integers

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n} = \frac{n-1}{m+n} I_{m,n-2}$$

$$(m \geqslant 2,\ n \geqslant 0) \qquad (n \geqslant 2,\ m \geqslant 0)$$

$$\int \tan^n\theta\,d\theta = \frac{1}{n-1}\tan^{n-1}\theta - \int \tan^{n-2}\theta\,d\theta\,;$$

$$\int \cot^n\theta\,d\theta = \frac{-1}{n-1}\cot^{n-1}\theta - \int \cot^{n-2}\theta\,d\theta.$$

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EXAMPLES V

1. Prove, by induction or otherwise, that the nth differential coefficient of $\tan\theta$ with respect to θ is expressible as a polynomial of degree $(n + 1)$ in $\tan\theta$, the powers being either all even or all odd.

   Express $\tan^5\theta$ as a linear function of $\tan\theta$ and its second and fourth derivatives. Use this to evaluate $\int \tan^5\theta d\theta$ and confirm your result by an independent determination of the integral.

2. (i) Evaluate $\displaystyle\int \frac{d\theta}{\sin\theta(1+\sin\theta)}$.

(ii) Prove that constants a, b, c can be found so that

$$a(\alpha x^2+1)+(bx+c)\frac{d}{dx}(\alpha x^2+1)\equiv x^2+1 \quad (\alpha>0).$$

Hence show that $\displaystyle\int \frac{x^2+1}{(\alpha x^2+1)^2}\,dx$ can be expressed as a rational function of x only if $\alpha=-1$ or 0.

3. Evaluate $\displaystyle\int \frac{dx}{x^3-x^2-9x+9}$.

Prove that

(i) $\displaystyle\int_0^{\pi}\frac{d\theta}{5+4\cos\theta}=\frac{\pi}{3}$, (ii) $\displaystyle\int_0^{\infty} e^{-x}\sin^2 x\,dx=\frac{2}{5}$.

4. Evaluate the integrals

$$\int\frac{dx}{6-x-4x^2-x^3},\quad \int\left(\frac{x}{c-x}\right)^{\frac{1}{2}}dx.$$

Prove that

$$\int_0^{\log_e 2}\frac{dx}{\sinh x+5\cosh x}=\frac{1}{\sqrt{6}}\{\tan^{-1}(\sqrt{6})-\tan^{-1}(\tfrac{1}{2}\sqrt{6})\}.$$

5. (i) Evaluate $\int \log x\,.\,dx$, and deduce that

$\int \sin\theta\log(1-\cos\theta)\,d\theta=(\cos\theta-\cos\theta)\log(e^{-1}-\cos\theta)$.

(ii) Given that $x^2+y^2=2ay$, use the substitution $y=tx$ to show that $\displaystyle\int\frac{dx}{y}=\frac{-x}{y}-2\tan^{-1}y/x$.

6. (*a*) Evaluate the following integrals :

(i) $\displaystyle\int\frac{5-7x}{2x^3-x^2-2x+1}dx$, (ii) $\displaystyle\int\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}d\theta$,

(*b*) Prove that $\displaystyle\int_{\frac{1}{2}}^{1}\sin^{-1}(\sqrt{x})dx=(\pi-1)/4$.

7. (*a*) Evaluate (i) $\displaystyle\int\frac{x\,dx}{(x-1)(x^2+1)}$, (ii) $\displaystyle\int\frac{dx}{1+\sin x+\cos x}$.

(*b*) Use the substitution $x=\tan\theta$ to evaluate

$$\int\frac{x^2(x+1)}{(x^2+1)^3}dx.$$

8. (i) Evaluate $\displaystyle\int\frac{5\cos\theta}{2\cos\theta+\sin\theta+2}\,d\theta$.

(ii) By considering the graph of $\sin x$, show that

$$\sin x\geqslant 2x/\pi \text{ for } 0\leqslant x\leqslant \pi/2.$$

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Use this to show that, as R increases without limit through positive values, the value of $R\int_0^{\frac{\pi}{2}} e^{-R\sin x}\,dx$ never exceeds $\pi/2$.

9. (*a*) Evaluate the integral $\int \frac{(x^2+2)dx}{x(x^2+4x+6)}$.

(*b*) Prove that (i) $\int_8^{15} \frac{dx}{(x-3)\sqrt{(x+1)}} = \frac{1}{2}\log_e \frac{5}{3}$.

(ii) $\int_{\frac{1}{4}}^{1} \frac{dx}{x\sqrt{(5x^2-4x+1)}} = \log_e(1+\sqrt{2})$.

10. Find the integrals (i) $\int (x-1)^{-1/2}(x-2)^{-1/2}dx$, (ii) $\int (1+x^2)^{-3/2}dx$,

(iii) $\int \sec^3 x\,dx$.

11. Find the indefinite integrals

(i) $\int \frac{dx}{x^2(a^2+x^2)}$, (ii) $\int \frac{dx}{(x^2+4x+5)^{1/2}}$,

(iii) $\int x\sin^2 x\,dx$, (iv) $\int \cosh^2 x\,dx$. (*I*)

12. Evaluate correctly to three significant figures

(i) $\int_2^3 \frac{dx}{\{(3-x)(x-2)\}^{1/2}}$, (ii) $\int_0^{\frac{\pi}{2}} x\sin 2x\cos x\,dx$,

(iii) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{8+17\cos\theta}$, (iv) $\int_0^1 x^2e^{-2x}\,dx$. (*I*)

13. (i) Prove that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$, and hence evaluate $\int_{\pi/n}^{(n-1)\pi/n} x\sin^3 x\,dx$.

(ii) Evaluate the integral $\int_{-1}^{+1} \frac{dx}{\sqrt{(1+x+x^2)}}$. (*I*)

14. Find the indefinite integrals

(i) $\int \cos^5\theta\,d\theta$, (ii) $\int x\sqrt{(1+x)}dx$,

(iii) $\int x^2e^{-\frac{1}{2}x}\,dx$, (iv) $\int \frac{du}{\sinh u}$. (*I*)

15. Evaluate correct to three significant figures

(i) $\int_0^{\frac{\pi}{4}} \tan\theta\,d\theta$, (ii) $\int_0^1 \frac{x^4dx}{1+x^2}$,

(iii) $\int_0^{\frac{\pi}{2}} \sin^3\theta(\sin^3\theta+\cos^3\theta)d\theta$. (*I*)

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16. Evaluate correct to three significant figures

$$\text{(i)} \int_0^1 x^3(1-x)^3 dx, \qquad \text{(ii)} \int_0^{\frac{\pi}{4}} \cos^3\theta\, d\theta,$$

$$\text{(iii)} \int_0^1 \frac{dx}{\sqrt{(2x-x^2)}}, \qquad \text{(iv)} \int_0^{\frac{\pi}{2}} \frac{\sin 3\theta}{\sin\theta}\, d\theta. \qquad (I)$$

17. Show that $\dfrac{d}{dx}\left\{\cot^{-1}\left(\dfrac{\cot\alpha\cos\alpha}{1+\operatorname{cosec}\alpha\sin x}\right)\right\} = \dfrac{\cos\alpha}{1+\sin\alpha\sin x}$.

Deduce the value of $\displaystyle\int \frac{dx}{A + B\sin x}$ $(B < A)$ (I)

18. (i) Evaluate $\displaystyle\int \frac{dx}{1+\sin x\cos x}$.

(ii) Evaluate $\displaystyle\int_0^{\pi} x\sin^6 x\cos^4 x\, dx$ by means of the substitution $x = \pi - t$. (II)

19. (*a*) Evaluate (i) $\displaystyle\int \frac{x-1}{(x+1)(x^2+1)} dx$, (ii) $\displaystyle\int_1^2 x^2\log x\, dx$,

(iii) $\displaystyle\int_0^{\frac{\pi}{4}} \sin^2\theta\cos^4\theta\, d\theta$.

(*b*) The density of a sphere of radius a at distance r from the centre is $k\left(1 - \dfrac{2r}{3a}\right)$. Prove that the mean density is half the density at the centre. (II)

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20. If $I_{m,n} = \displaystyle\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\, d\theta$, where m, n are positive integers, prove that $(m+n)I_{m,n} = (m-1)I_{m-2,n}$.

Evaluate $\displaystyle\int_0^1 x^2(1-x^2)^{5/2} dx$.

21. If $I_n = \displaystyle\int_0^x \frac{t^n}{\sqrt{(1+t+t^2)}} dt$, prove that

$$(n+1)I_{n+1} + (n+\tfrac{1}{2})I_n + n\,I_{n-1} = x^n\sqrt{(1+x+x^2)}.$$

Hence, or otherwise, prove that

$$\int_0^1 \frac{x^3}{\sqrt{(1+x+x^2)}} dx = \left\{21\log_e\left(1+\frac{2\sqrt{3}}{3}\right) - 2(3\sqrt{3}-1)\right\}/48.$$

22. Obtain a reduction formula expressing $\int (x^2+a^2)^{\frac{1}{2}n}\, dx$ in terms of $\int (x^2+a^2)^{\frac{1}{2}n-1}\, dx$.

Prove that $\displaystyle\int_0^a (x^2+a^2)^{\frac{3}{2}}\, dx = \tfrac{1}{8}a^4\{7\sqrt{2} + 3\log(1+\sqrt{2})\}$.

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23. If $I_{m,\,n} = \int \cos^m x \sin nx\, dx$, prove that

$$(m+n)I_{m,\,n} = -\cos^m x \cos nx + mI_{m-1,\,n-1}.$$

Hence, or otherwise, prove that

$$\int_0^{\frac{\pi}{4}} \cos^2 x \sin 4x\, dx = 5/12.$$

24. If $V_n = \int_0^x \frac{dt}{(t^2+a^2)^n}$ and $W_{m,\,n} = \int_0^x \frac{t^m}{(t^2+a^2)^n}\, dt$, where m, n are positive integers, obtain the reduction formulae

$$V_{n+1} = \frac{1}{2na^2}\frac{x}{(x^2+a^2)^n} + \frac{2n-1}{2na^2}V_n,$$

$$W_{m,\,n} = \frac{-1}{2(n-1)}\frac{x^{m-1}}{(x^2+a^2)^{n-1}} + \frac{m-1}{2(n-1)}W_{m-2,\,n-1}.$$

Evaluate $W_{4,\,4}$.

25. If $I_{m,\,n} = \int \sin^m \theta \cos^n \theta\, d\theta$, where m, n are integers greater than unity, prove that

$$(m+n)I_{m,\,n} = \sin^{m+1}\theta \cos^{n-1}\theta + (n-1)I_{m,\,n-2}$$
$$= -\sin^{m-1}\theta \cos^{n+1}\theta + (m-1)I_{m-2,\,n}.$$

Hence, or otherwise, evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \sin^{2p}\theta \cos^{2q}\theta\, d\theta$$

where p, q are positive integers.

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26. If $I_n = \int \frac{\sin(2n-1)x}{\sin x}\, dx$, $J_n = \int \frac{\sin^2 nx}{\sin^2 x}\, dx$, prove that, apart from constants of integration,

$$n(I_{n+1} - I_n) = \sin 2nx,$$
$$J_{n+1} - J_n = I_{n+1}.$$

Evaluate I_3 and J_3. *(II)*

27. If $I_n = \int e^{ax} \cos^n x\, dx$, where n is a positive integer, prove that

$$I_n = \frac{1}{a^2+n^2}e^{ax}\cos^{n-1}x(a\cos x + n\sin x) + \frac{n(n-1)}{a^2+n^2}I_{n-2}.$$

Hence evaluate $\int_0^{\frac{\pi}{2}} e^x \cos^4 x\, dx$. *(II)*

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CHAPTER VI

Power Series, Maxima, Minima, and Points of Inflexion

1. Maclaurin's Theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + \ldots$$

Taylor's Theorem (x and h independent):

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \ldots + \frac{x^n}{n!}f^{(n)}(h) + \ldots$$

$$= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots + \frac{h^n}{n!}f^{(n)}(x) + \ldots$$

2. Conditions for turning points (stationary) and points of inflexion

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Type of point	dy/dx	d^2y/dx^2
Maximum	0	negative
Minimum	0	positive
Point of inflexion	any value	0

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EXAMPLES VI

1. State the Taylor's expansion of the function $f(x + a)$ in ascending powers of x.

   Find the smallest positive root of the equation $4 \sin^2 x = 3 + \alpha$, where α^3 and higher powers of α may be neglected.

2. If $f(x)$ can be expanded in a power series $\sum_{n=0}^{\infty} a_n x^n$, prove that $n!\, a_n = f^{(n)}(0)$ where $f^{(n)}(x)$ is the nth derivative of $f(x)$.

   Assuming the existence of the expansion, show that

$$2 \cos x \cosh x = \sum_{n=0}^{\infty} 2^{\frac{1}{2}n} \{1 + (-1)^n\} \frac{x^n}{n!} \cos \frac{n\pi}{4}.$$

3. If $f(x)$ possesses a Maclaurin's series and $f(x) = f(-x)$, prove that the series contained only even powers of x.

   Expand $x \operatorname{cosec} x$ as far as the fourth power of x inclusive.

4. Prove that no odd powers of x occur in the expansion in ascending powers of x, of the function $\dfrac{x(e^x + 1)}{2(e^x - 1)}$.

   By means of Maclaurin's theorem, or otherwise, expand the function as far as the term containing x^4.

   Hence, or otherwise, prove that

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{1}{6}\frac{x^2}{2!} - \frac{1}{30}\frac{x^4}{4!} + \ldots$$

5. Show that $\tan\left(\dfrac{\pi}{4} + x\right) = 1 + 2x + 2x^2 + \dfrac{8}{3}x^3 + \dfrac{10}{3}x^4 + \ldots$

   Evaluate $\tan(46^\circ . 30')$ correct to four decimals. (*I*)

6. Prove that $\dfrac{d}{dx}\left(\tan^{-1}\dfrac{x}{a}\right) = \dfrac{a}{a^2 + x^2}$.

   Hence, or otherwise, find the expansion of $\tan^{-1}(x/a)$ in (i) ascending powers of x, (ii) ascending powers of $1/x$.

   For what range of values of x are these expansions valid? (*I*)

7. Prove that $\dfrac{d}{dx}\{\log_e(1 + x)\} = \dfrac{1}{1 + x}$, and hence, by assuming that $x^2 < 1$, expand $\log_e(1 + x)$ in ascending powers of x.

   Obtain a series in ascending powers of x for $\log_e \dfrac{1 + x}{1 - x}$ and use the expansion to calculate $\log_e 11/9$ correct to three places of decimals. (*I*)

8. The curve whose equation is $y = ax^3 + bx^2 + cx + d$, passes through the points (1, 3), (0, 1) and has a tangent parallel to the x-axis at the point (2, 1). Prove that the maximum and minimum points on the curve are at the points where $x = 2/3$ and $x = 2$ respectively. Draw a sketch of the graph. (*I*)

9. (i) Show that the gradient of the curve whose equation is $y = e^{-1/x}$ is positive for all values of x and draw a rough sketch of the curve.

   (ii) Two sources of light, A and B, of intensities a and b respectively are at a distance d apart. Find the distance from A of the point of the line AB at which the illumination is least.

   (Assume that the illumination is proportional to the intensity and inversely proportional to the square of the distance.) (*I*)

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10. A strip of sheet metal 40 inches wide is to be bent to form an open channel of trapezoidal section, the base width being 20 inches. At what angle should the sides slope in order that the sectional area may be a maximum? *(I)*

11. (i) Find the maxima and minima of $2\sin^2 x - 3(1 + \cos x)^2$ in the range $0 \leqslant x < 360°$.

    (ii) Find to the nearest degree the angle of the sector to be cut from a given circular sheet in order that the remainder may be bent to form the right circular cone of maximum volume. *(I)*

12. (*a*) Differentiate $\log\left\{\sin^{-1}\left(\frac{1-x}{1+x}\right)\right\}$ with respect to x.

    (*b*) The total perimeter of a sector of a circle is given. Find the angle of the sector when the area is greatest, giving your answer in degrees. *(I)*

13. (*a*) If $y = (2x + 3)^4 \sec^3 (x - 2)e^{3x+4}$, evaluate $\frac{1}{y}\frac{dy}{dx}$.

    (*b*) If a function $f(x)$ consist of the sum of two positive terms, one of which is proportional to x^3 and the other inversely proportional to x^2, show that the function has its least value when the parts are equal. If $f(x) = 8$ when $x = 1$, and $f'(x) = 15$ when $x = 2$, draw a rough graph of the function. *(I)*

14. Points D on AB and E on AC are taken such that the area of the triangle ADE is $1/n$th of the area of the triangle ABC, where n is a given number. Find the positions of D and E if the length of DE is as small as possible. *(II)*

15. A ladder is to be taken in a horizontal position round a right-angled corner from a passage of width a to a passage of width b. Show that the greatest possible length of the ladder is $(a^{2/3} + b^{2/3})^{3/2}$. *(II)*

16. Find the minimum area of the triangle formed by the co-ordinate axes and a tangent to the curve whose parametric equations are

$$x = a\cos\theta, \quad y = a\sin^2\theta. \qquad (II)$$

17. By considering the derivative of the function

$$f(x) = \sin x \tan x - 2\log(\sec x)$$

prove that $f(x)$ steadily increases as x increases from 0 to $\pi/2$. Show also that the graph of the function has no inflexion between these limits.

Show that the function $2\sin x \tan x - 5\log(\sec x)$ has one minimum value in the range $0 < x < \pi/2$.

18. Sketch the graph of the curve $y = 4\cos x - 3\cos 2x$ from $x = 0$ to $x = \pi$.

Find the values of x giving maximum and minimum values of the function $y = 4\alpha\cos x - \beta\cos 2x$, where α, β are positive numbers. Distinguish between the cases (i) $\alpha > \beta$, (ii) $\alpha = \beta$, (iii) $\alpha < \beta$.

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19. P is a variable point on a parabola of latus rectum $4a$ and vertex O. The ordinate at P meets the axis at M, and Q is the foot of the perpendicular from M to OP. Find the length OM when QP − OQ is a maximum.

20. Show how to find the maximum and minimum values of a function $f(x)$ and give, with proof, a method of discriminating between them.

If $y^3 = x^2(6x - x^2 - 8)$, prove that the stationary values of y occur at $x = \frac{1}{4}(9 \pm \sqrt{17})$, and determine which of these gives a maximum value of y and which a minimum.

21. The equation of a plane curve is $y^3 + x^3 - 9xy + 1 = 0$ and (x_1, y_1) is a point on the curve at which the tangent is parallel to the x-axis. Prove that, at (x_1, y_1), $d^2y/dx^2 = 18/(27 - x_1^3)$.

Prove also that the stationary values of y occur at the points for which $x = (27 \pm 3\sqrt{78})^{1/3}$ and determine which of these gives a maximum value of y and which a minimum.

22. Find the maximum and minimum values of the function

$$t^3 - 3m^2t + 2n,$$

where m and n are real constants.

Hence prove that the equation $t^3 - 3m^2t + 2n = 0$ has three unequal real roots or only one real root according as m^6 is greater or less than n^2.

Prove that the curve whose equation is

$$y(y^2 - 3) = x(x^2 - 12)$$

is such that there are two points on it at which the tangents are parallel to the x-axis and six points at which the tangents are parallel to the y-axis.

23. Show how to find the maximum and minimum values of a function of one variable and how to distinguish between them.

Show that the distance r between a point P on the curve $x = 2a\cos^3 t$, $y = 2a\sin^3 t$ and the point (a, o) is least when the parameter of the point P is given by $8\cos t = \sqrt{33} + 1$.

Draw a rough graph showing the variation of r with t as the point P completely describes the curve.

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CHAPTER VII

Tangents, Normals, Curvature, Partial Differentiation, etc.

1. Tangents and normals

$\left(\frac{dy}{dx}\right)_1$ is the slope of the tangent at (x_1, y_1) on the curve $y = f(x)$.

Equation of tangent at (x_1, y_1) is $y - y_1 = \left(\frac{dy}{dx}\right)_1 (x - x_1)$;

Equation of normal at (x_1, y_1) is $(x - x_1) + \left(\frac{dy}{dx}\right)_1 (y - y_1) = 0$.

$\dot{y}/\dot{x}$ is the slope of the tangent at the point (t) on the curve

$$x = f_1(t), \quad y = f_2(t).$$

Equation of tangent at the point (t) is

$$\dot{x}[y - f_2(t)] = \dot{y}[x - f_1(t)];$$

Equation of normal at the point (t) is

$$\dot{x}[x - f_1(t)] + \dot{y}[y - f_2(t)] = 0.$$

2. Subtangent at (x, y) is $y \Big/ \frac{dy}{dx}$;

Subnormal at (x, y) is $y\, dy/dx$.

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}; \quad \frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2};$$

$$\frac{dy}{dx} = \tan\psi; \quad \frac{dx}{ds} = \cos\psi; \quad \frac{dy}{ds} = \sin\psi;$$

where s = length of arc of curve.

3. If (r, θ) be the polar co-ordinates of a point P and ϕ the angle the tangent to the curve makes with OP, then

$$\tan\phi = r\frac{d\theta}{dr}; \quad \cot\phi = \frac{1}{r}\frac{dr}{d\theta};$$

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$$\sin\phi = r\frac{d\theta}{ds}; \quad \cos\phi = \frac{dr}{ds}.$$

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2; \quad \left(\frac{ds}{dr}\right)^2 = r^2\left(\frac{d\theta}{dr}\right)^2 + 1.$$

4. Polar Subtangent $= r^2 d\theta/dr$;

Polar Subnormal $= dr/d\theta$.

5. Curvature $= d\psi/ds$;

Radius of curvature $\rho = ds/d\psi$

$$= \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2} \Big/ \frac{d^2y}{dx^2}$$

$$= \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{3/2} \Big/ \left(r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right)$$

$$= (\dot{x}^2 + \dot{y}^2)^{3/2}/(\dot{x}\ddot{y} - \ddot{x}\dot{y})$$

$$= r\,dr/dp,$$

where p is the perpendicular from O on the tangent.

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6. If $(\bar{x}, \bar{y})$ be the centre of curvature for point (x, y), then

$$\bar{x} = x - \frac{dy}{dx}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\} \Big/ \frac{d^2y}{dx^2};$$

$$\bar{y} = y + \left\{1 + \left(\frac{dy}{dx}\right)^2\right\} \Big/ \frac{d^2y}{dx^2}.$$

7. Partial Differentials

If $u, v, w, \ldots$ are functions of $x, y, z, \ldots$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} + \ldots \\ \frac{\partial u}{\partial y} &= \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y} + \ldots \end{aligned}\right\}, \text{ where } u = v + w + \ldots \text{ etc.}$$

$$\frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x};$$

$$\frac{\partial}{\partial y}(uv) = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}, \text{ etc.}$$

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$$\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\,\partial u/\partial x - u\,\partial v/\partial x}{v^2};$$

$$\frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{v\,\partial u/\partial y - u\,\partial v/\partial y}{v^2}, \text{ etc.}$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{du}{dp}\cdot\frac{\partial p}{\partial x}; \\ \frac{\partial u}{\partial y} &= \frac{du}{dp}\cdot\frac{\partial p}{\partial y}, \text{ etc.} \end{aligned}\right\} \quad u \text{ is a function of } p, \text{ where } p \text{ is a function of } x, y, z, \ldots$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}, \text{ where } u \text{ is a continuous function of } x, y.$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}\cdot dy + \frac{\partial u}{\partial z}\cdot dz + \ldots;$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt} + \ldots,$$

where $x, y, z, \ldots$ are all functions of t.

For the curve $f(x, y) = 0$,

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}.$$

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If $x, y, z, \ldots$ be all functions of several variables $w_1, w_2, \ldots$

$$\frac{\partial u}{\partial w_1} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial w_1} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial w_1} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial w_1} + \ldots;$$

$$\frac{\partial u}{\partial w_2} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial w_2} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial w_2} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial w_2} + \ldots, \text{ etc.}$$

EXAMPLES VII

1. A curve is given by the parametric equations

$$x = 1/(1 + t^2), \quad y = t^3/(1 + t^2).$$

Find the equation of the tangent to the curve at the point whose parameter is t, and show that the area of the triangle formed by the tangent and the co-ordinate axes is not greater than $(3\sqrt{3})/8$ square units. Sketch the curve.

2. Find the equation of the normal at the point $T(at^2, 2at)$ of the parabola $y^2 = 4ax$.

A segment TP of constant length l is measured off in the outward section along the normal. Find, in terms of the parameter t, the equations of the locus of P. *(I)*

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3. For the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$, prove that the length of the perpendicular from the origin on to the tangent at any point is $a\sin\theta\cos\theta$, and that the radius of curvature is $3a\sin\theta\cos\theta$. (*I*)

4. If (X, Y) is the centre of curvature at the point (x, y) of a plane curve, show that

$$X = x - y'(1 + y'^2)/y'', \quad Y = y + (1 + y'^2)/y''.$$

Find the equation to the locus of the centre of curvature of $y^2 = 4ax$. (*I*)

5. If $x = a\sin 2\theta$, $y = a\cos\theta$, prove that

$$\frac{d^2y}{dx^2} = -\frac{1}{4a}\,\frac{\cos\theta(1 + 2\sin^2\theta)}{\cos^3 2\theta}.$$

Sketch the curve given by the above values of x and y and find its radius of curvature at the point (o, a). (*I*)

6. Prove the formula $\dfrac{1}{\rho} = \dfrac{d^2y}{dx^2} \Big/ \left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}$ for the radius of curvature ρ of a plane curve.

Find the radius of curvature and the co-ordinates of the centre of curvature at the point $(\frac{1}{2}, \frac{5}{8})$ on the curve

$$y = x^3 + 2x^2 - 2x + 1. \qquad (I)$$

7. Prove that the centre of curvature at the point $(a\cos\theta, b\sin\theta)$ on the ellipse $x^2/a^2 + y^2/b^2 = 1$ is

$$\left(\frac{a^2 - b^2}{a}\cos^3\theta, \quad \frac{b^2 - a^2}{b}\sin^3\theta\right).$$

Deduce the equation of the locus of the centre of curvature. (*I*)

8. Define *curvature* and *radius of curvature* at any point of a plane curve.

If $x = e^t(\cos t + \sin t)$, $y = e^t(\cos t - \sin t)$, find the equation of the tangent at the point (x_0, y_0) where $t = t_0$. Find the radius of curvature at this point and prove that the co-ordinates of the centre of curvature are $(y_0, -x_0)$.

9. Sketch the graph of the plane curve $x = \sin^3\theta$, $y = 3\cos^3\theta$ for values of θ lying in the range 0 to 2π.

P is the point on the curve for which $\theta = \pi/3$ and C is the centre of curvature at P. Show that the equation of the line CP is $y\sqrt{3} = x$ and find the length of CP.

10. Prove that the curve represented by the equation $y^3 = x(x + 2y)$ possesses a minimum ordinate when $x = 1$, and show that the radius of curvature of the curve at this minimum point is $1/2$.

Prove also that the locus of the mid-points of chords parallel to the x-axis is the straight line $x + y = 0$, and sketch clearly on the (x, y)-plane the general shape of the curve.

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11. The position of a point P on a curve is defined by the radius vector r drawn from the origin and making an angle θ with the initial line. The perpendicular distance from the origin to the tangent at P being denoted by p, prove that $1/p^2 = u^2 + (du/d\theta)^2$, where u is the reciprocal of r.

Hence, or otherwise, show that the radius of curvature at P is given by $(u^2 + u'^2)^{3/2}/u^3(u + u'')$, where accents denote differentiations with respect to θ.

12. Prove that the centre of curvature at the point P $(at^2, 2at)$ on the parabola $y^2 = 4ax$ has co-ordinates

$$x = 2a + 3at^2, \quad y = -2at^3.$$

A segment PQ of length a is measured along the tangent to the parabola at the point P in the direction of t increasing. Prove that the equation of the locus of Q is

$$x = a\left\{t^2 + \frac{t}{\sqrt{(1+t^2)}}\right\}, \quad y = a\left\{2t + \frac{1}{\sqrt{(1+t^2)}}\right\},$$

and prove that the normal at Q to this locus passes through the centre of curvature at the point P on the parabola.

13. Define the curvature at a point on a plane curve and obtain a formula for the curvature if the rectangular cartesian equation of the curve, $y = f(x)$, is given.

A, A′; B, B′ are the ends of the major and minor axes of an ellipse of centre O and C is the fourth vertex of the rectangle whose other vertices are A, O, B. Prove that the line through C perpendicular to AB cuts AA′ in the centre of curvature at A and BB′ in the centre of curvature at B.

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14. At the point P(x, y) on a plane curve, the centre of curvature is C (X, Y) and the inclination of the tangent to the x-axis is ψ. Prove that $\quad X = x - dy/d\psi, \quad Y = y + dx/d\psi.$

P is the point (x, y) on the curve $x = a\cos^3(\phi/3)$, $y = a\sin^3(\phi/3)$ and C is the centre of curvature corresponding to P. A point N divides CP internally so that CN = 2NP. Find the locus of N.

15. Prove that the radius of curvature ρ of a plane curve is given in cartesian co-ordinates by

$$\frac{1}{\rho} = \frac{d^2y}{dx^2} \Big/ \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}.$$

If $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$, prove that $\rho = at$. (*I*)

16. A plane curve has the polar equation $r = f(\theta)$, prove that the perpendicular from the pole to the tangent at the point (r, θ) is given by

$$1/p^2 = 1/r^2 + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$$

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Find the (p, r) equation of the curve $r^4 = a^4 \cos 4\theta$.

If M be the foot of the perpendicular from the pole O to the tangent at P, prove that the maximum value of the area of the triangle OPM occurs when P is the point for which $\cos 4\theta = \sqrt{3}/5$.

17. With the usual polar co-ordinate notation prove that

$$\cot \phi = \frac{1}{r}\frac{dr}{d\theta}.$$

The limaçon $r = a - b \cos \theta$ $(a > b)$ is cut at the points A and B by a circle touching the line $\theta = 0$ at the origin. Prove that the tangents to the limaçon at A and B intersect at a point on the circle.

18. A point moves so that the product of its distances from two fixed points, A and B, is constant and equal to $3c^2$, where $c = \frac{1}{2}$AB. By taking the mid-point of AB as pole, show that the polar equation of the locus of the point can be expressed in the form

$$r^4 - 2c^2r^2 \cos 2\theta - 8c^4 = 0.$$

Find the (p, r) equation of the curve, and hence show that its curvature at any point is $(3r^4 - 8c^4)/6c^2r^3$.

19. The polar equation of a plane curve is $r = f(\theta)$. OP is the radius-vector to the point P on the curve and PT, PN are respectively the tangent and normal to the curve at P.

If a perpendicular to OP through O intersects PT at A and PN at B, prove that $\mathrm{OB/OA} = \dfrac{1}{r^2}\left(\dfrac{dr}{d\theta}\right)$.

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A curve is such that (i) $\mathrm{OB/OA} = \theta^2$; (ii) r is infinity when θ = infinity, and (iii) $r = a$ when $\theta = 0$. Find the equation of the curve and prove that the area of the triangle APB is $\frac{1}{2}a^2e\theta^2\left(\theta + \dfrac{1}{\theta}\right)$.

20. If $u = f(x, y)$ and δu, δx, δy denote small changes in u, x, y respectively, show that approximately

$$\delta u = \frac{\partial u}{\partial x} \cdot \delta x + \frac{\partial u}{\partial y} \cdot \delta y.$$

The side c of a triangle is determined by measuring the sides a, b, and the angle C. If the possible errors of measurement are δa, δb and δC respectively, show that

$$\delta c = \cos \mathrm{B} \cdot \delta a + \cos \mathrm{A} \cdot \delta b + a \sin \mathrm{B} \cdot \delta \mathrm{C}.$$

How should the triangle be chosen so as to minimise the effect of the error made in the measurement of the side a?

21. If $z = (x + y)\phi(y/x)$, where ϕ is an arbitrary function, prove that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z,$$

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and that $$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=0.$$

22. If $\phi(z)=y/x$, prove that

$$\frac{\partial z}{\partial x}+\phi(z)\frac{\partial z}{\partial y}=0,$$

and that $$\left(\frac{\partial z}{\partial y}\right)^2\frac{\partial^2 z}{\partial x^2}+\left(\frac{\partial z}{\partial x}\right)^2\frac{\partial^2 z}{\partial y^2}=2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\cdot\frac{\partial^2 z}{\partial x\partial y}.$$

23. If $u=x^n\,\mathrm{F}(x/y)$, where F denotes an arbitrary function, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu,$$

and hence that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.$$

24. (i) If the variables x, y, z are connected by the equations $f(x, y, z)=0$, $x^2+y^2+z^2=$ constant, prove that

$$\frac{dy}{dz}=-\left(z\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial z}\right)\Big/\left(z\frac{\partial f}{\partial y}-y\frac{\partial f}{\partial z}\right).$$

(ii) If $z=f(x+y)g(x-y)$, where f and g are arbitrary functions, prove that

$$z\frac{\partial^2 z}{\partial x^2}-z\frac{\partial^2 z}{\partial y^2}=\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2.$$

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25. (i) If V be a function of x and y, and $x=r\cos\theta$, $y=r\sin\theta$, prove that

$$\frac{\partial^2 \mathrm{V}}{\partial x^2}+\frac{\partial^2 \mathrm{V}}{\partial y^2}=\frac{\partial^2 \mathrm{V}}{\partial r^2}+\frac{1}{r}\frac{\partial \mathrm{V}}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \mathrm{V}}{\partial \theta^2}.$$

(ii) If $\mathrm{U}=y\log(y+r)-r$, where $r^2=x^2+y^2$, prove that

$$\frac{\partial^2 \mathrm{U}}{\partial x^2}+\frac{\partial^2 \mathrm{U}}{\partial y^2}=\frac{1}{y+r}.$$

26. (i) If $f(x, y)\equiv \mathrm{F}(u, v)$, where $u=x^2-y^2$ and $v=2xy$, show that

$$\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2=4(u^2+v^2)^{\frac{1}{2}}\left[\left(\frac{\partial \mathrm{F}}{\partial u}\right)^2+\left(\frac{\partial \mathrm{F}}{\partial v}\right)^2\right],$$

and $$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=4(u^2+v^2)\left[\frac{\partial^2 \mathrm{F}}{\partial u^2}+\frac{\partial^2 \mathrm{F}}{\partial v^2}\right].$$

(ii) If x, y and z satisfy the relations $f(x, y, z)=$ constant and $xyz=$ constant, show that

$$\frac{dy}{dx}=-\frac{y}{x}\left(x\frac{\partial f}{\partial x}-z\frac{\partial f}{\partial z}\right)\Big/\left(y\frac{\partial f}{\partial y}-z\frac{\partial f}{\partial z}\right).$$

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27. Prove that, if V is a homogeneous function in x and y of degree n

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = nV.$$

If in addition $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$ and $u = \frac{V}{r^{2n}}$, where $r^2 = x^2 + y^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

28. (i) If $z = f(x, y) + g(u)$, where $u = xy$ and f and g are arbitrary functions, show that $w = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$ is independent of the choice of g.

Find w when $f(x, y) = xye^{x-y}$.

(ii) Show that at the point (a, a) on the curve $x^3 + y^3 - axy = a^3$ the value of $\frac{d^2y}{dx^2}$ is $-7/a$.

29. If $z^2 = x^2 + y^2 + 1$, show that

(i) $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.

(ii) $xy\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right) = (x^2 - y^2)\frac{\partial^2 z}{\partial x \partial y}$. *(II)*

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30. If z be a function of x and y, and these are functions of u and v defined by the equations $x = uv$, $y = (u + v)/(u - v)$, show that

$$2x\frac{\partial z}{\partial x} = u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v},$$

$$2y\frac{\partial z}{\partial y} = \tfrac{1}{2}(u^2 - v^2)\left\{\frac{1}{u}\frac{\partial z}{\partial v} - \frac{1}{v}\frac{\partial z}{\partial u}\right\}.$$ *(II)*

31. (i) If $u = x/r^3$ and $r^2 = x^2 + y^2 + z^2$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

(ii) If x increase at the rate of two inches per second at the instant when $x = 3$ inches and $y = 1$ inch, at what rate must y be changing in order that the function $2xy - 3x^2y$ shall be neither increasing nor decreasing when x and y have these values? *(II)*

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CHAPTER VIII

Determinants

1. Definitions, etc.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \text{ is second order determinant.}$$

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ is a third order determinant whose value is } \begin{aligned} & a_1A_1 - b_1B_1 + c_1C_1 \\ &= a_1A_1 - a_2A_2 + a_3A_3, \end{aligned}$$

where A_1 is the minor of a_1 and $A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, B_1 is the minor of b_1 and $B_1 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$, etc.

2. Theorems

If two rows (or two columns) of a determinant be interchanged, the determinant changes sign.

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If two rows (or two columns) of a determinant be the same, the value of the determinant is zero.

If the elements of any row (or column) of a determinant be each multiplied by the same factor, the result is the product of that factor and the original determinant.

If each element of any row (or column) of a determinant consists of the algebraic sum of r terms, the determinant is equivalent to the sum of r other determinants in each of which the elements consist of single terms.

If the elements of any row (or column) be increased or diminished by equimultiples of the corresponding elements of any other row (or column), the value of the determinant is unaltered.

3. The solution of the equations

$$\begin{aligned} a_1x + b_1y + c_1z + k_1 &= 0, \\ a_2x + b_2y + c_2z + k_2 &= 0, \\ a_3x + b_3y + c_3z + k_3 &= 0, \end{aligned}$$

is

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{-1}{\Delta_0},$$

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where $\Delta_1 = \begin{vmatrix} b_1 & c_1 & k_1 \\ b_2 & c_2 & k_2 \\ b_3 & c_3 & k_3 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a_1 & c_1 & k_1 \\ a_2 & c_2 & k_2 \\ a_3 & c_3 & k_3 \end{vmatrix}$,

$$\Delta_3 = \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}, \quad \Delta_0 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

4. If the equations $a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$, $a_3x + b_3y + c_3z = 0$ are consistent, then the condition (eliminant) is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

EXAMPLES VIII

1. (i) Factorise the expression $\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}$.

(ii) Find the general value of θ that satisfies the equation

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 2\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 2\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 2\theta \end{vmatrix} = 0.$$

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2. If the corresponding elements of two rows of a three-row determinant are equal, prove that the value of the determinant is zero.

Prove that the value of the determinant

$$\begin{vmatrix} \cos\theta & (1+\tfrac{1}{2}\sec\theta)\sec^2(\theta/2) & \sin\theta \\ \cos^2\theta & 1 & \sin\theta\cos\theta \\ 2\sin\theta\cos^2(\theta/2) & C & \sin^2\theta - \cos\theta \end{vmatrix}$$

is $\frac{1}{2}\cos\theta\sec^2(\theta/2)$.

3. The two values of y given by the simultaneous equations

$$y = \frac{a_1x^2 + b_1x + c_1}{a_2x^2 + b_2x + c_2} \quad \text{and} \quad ax^2 + bx + c = 0$$

are equal.

Prove that *either* $b^2 = 4ac$ *or* $\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$

4. Prove that $\begin{vmatrix} a+x & a_1 & b_1 \\ b+y & a_2 & b_2 \\ c+z & a_3 & b_3 \end{vmatrix} = \begin{vmatrix} a & a_1 & b_1 \\ b & a_2 & b_2 \\ c & a_3 & b_3 \end{vmatrix} + \begin{vmatrix} x & a_1 & b_1 \\ y & a_2 & b_2 \\ z & a_3 & b_3 \end{vmatrix}$

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Express the determinant

$$\begin{vmatrix} 2 & a+\alpha+b+\beta & ab+\alpha\beta \\ a+\alpha+b+\beta & 2(a+b)(\alpha+\beta) & ab(\alpha+\beta)+\alpha\beta(a+b) \\ ab+\alpha\beta & ab(\alpha+\beta)+\alpha\beta(a+b) & 2\alpha\beta ab \end{vmatrix}$$

as the sum of eight determinants, and hence, or otherwise, show that its value is zero.

5. (i) Factorise $\begin{vmatrix} x+y+nz & (n-1)x & (n-1)y \\ (n-1)z & y+z+nx & (n-1)y \\ (n-1)z & (n-1)x & z+x+ny \end{vmatrix}$.

(ii) Prove that $\begin{vmatrix} \cos(x+\alpha) & \sin(x+\alpha) & 1 \\ \cos(x+\beta) & \sin(x+\beta) & 1 \\ \cos(x+\gamma) & \sin(x+\gamma) & 1 \end{vmatrix}$

is independent of x.

6. Prove that if, in a third-order determinant, the corresponding elements of any two columns are identical the determinant is zero.

(i) Factorise the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ (b+c)^2 & (c+a)^2 & (a+b)^2 \end{vmatrix}.$$

(ii) Solve the equation $\begin{vmatrix} x+1 & 2x & 1 \\ x & 3x-2 & 2x \\ 1 & x & x \end{vmatrix} = 0.$

7. (i) If Δ is a determinant of the third order, whose elements are polynomials in x, and if all the rows of Δ become identical when x takes the value α, prove that $(x-\alpha)^2$ is a factor of Δ.

(ii) Factorise the determinants

$$(a)\ \begin{vmatrix} 1 & x & x^2 \\ 1 & x^2 & x^4 \\ 1 & x^3 & x^6 \end{vmatrix}, \quad (b)\ \begin{vmatrix} \beta^2\gamma^2+\alpha^2\delta^2 & \beta\gamma+\alpha\delta & 1 \\ \gamma^2\alpha^2+\beta^2\delta^2 & \gamma\alpha+\beta\delta & 1 \\ \alpha^2\beta^2+\gamma^2\delta^2 & \alpha\beta+\gamma\delta & 1 \end{vmatrix}.$$

8. Factorise the determinants

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}, \qquad \begin{vmatrix} 2 & a+b & a^2+b^2 \\ a+b & a^2+b^2 & a^3+b^3 \\ 1 & c & c^2 \end{vmatrix}.$$

Hence, or otherwise, show that

$$\begin{vmatrix} 2 & a+b & a^2+b^2 \\ a+b & a^2+b^2 & a^3+b^3 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ a & b & 0 \\ 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}.$$

9. (i) Express $\begin{vmatrix} x^2 & y^2 & z^2 \\ yz & zx & xy \\ 1 & 1 & 1 \end{vmatrix}$ as a product of three linear factors.

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(ii) If no two numbers, a, b, c are equal, find the condition that the equations

$$x + y + z = 0, \quad x/a + y/b + z/c = 0, \quad a^2x + b^2y + c^2z = 0$$

may have a consistent set of non-zero solutions. Find the ratios $x : y : z$ when $a = 1$, $b = -3$, and $c = 2$.

10. (i) Prove that, if to any row of a given determinant of the third order be added a fixed multiple of another row, the value of the determinant is unaltered.

(ii) Solve the equations

$$\begin{aligned} ax + by + cz &= 1, \\ a^2x + b^2y + c^2z &= 1, \\ a^4x + b^4y + c^4z &= 1, \end{aligned}$$

simplifying the results as far as possible.

11. (*a*) Solve the equation $\begin{vmatrix} 15 - 2x & 11 & 10 \\ 11 - 3x & 17 & 16 \\ 7 - x & 14 & 13 \end{vmatrix} = 0.$

(*b*) Express as a determinant the result of eliminating x, y, z from the equations

$$x + y + z = 0, \quad ax + by + cz = 0, \quad a^2x + b^2y + c^2z = 0,$$

and deduce the relations between a, b, c which are necessary for the eliminations to be necessary. (*II*)

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12. Eliminate x from the equations

$$x^2 + ax + b = 0, \quad x^2 + px + q = 0.$$

Show that the eliminant is equivalent to

$$\begin{vmatrix} 1 & a & b & 0 \\ 0 & 1 & a & b \\ 0 & 1 & p & q \\ 1 & p & q & 0 \end{vmatrix} = 0. \qquad (II)$$

13. Find the values of λ for which the equations

$$\begin{aligned} (2 - \lambda)x + 2y + 3 &= 0, \\ 2x + (4 - \lambda)y + 7 &= 0, \\ 2x + 5y + 6 - \lambda &= 0, \end{aligned}$$

are consistent, and find values of x and y corresponding to each of these values of λ.

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CHAPTER IX

Straight Line, Circle, and Parabola

Change of origin and axes

1. For the curve $y = f(x)$, if the origin be moved to the point (h, k) the equation of the curve becomes $Y + k = f(X + h)$, where X, Y are the new co-ordinates.

If the axes be turned through an angle θ, and X, Y be the new co-ordinates, the equation of the curve $y = f(x)$ becomes

$$X \sin\theta + Y \cos\theta = f(X \cos\theta - Y \sin\theta).$$

Straight lines

2. The lines represented by

$$a_1x + b_1y + c_1 = 0,$$
$$a_2x + b_2y + c_2 = 0,$$
$$a_3x + b_3y + c_3 = 0,$$

are concurrent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

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Any straight line through the point of intersection of the straight lines

$$a_1x + b_1y + c_1 = 0, \quad a_2x + b_2y + c_2 = 0$$

is given by $a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$, where λ is any constant.

If θ be the angle between the straight lines given by

$$ax^2 + 2bxy + cy^2 = 0,$$

then

$$\theta = \tan^{-1} \pm \frac{2\sqrt{(b^2 - ac)}}{a + c}.$$

The condition that $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ shall represent two straight lines is

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \text{i.e.} \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

The equation of the pair of straight lines bisecting the angle between the two lines given by $ax^2 + 2hxy + by^2 = 0$, is

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}.$$

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The Circle

3. The polar of (h, k) with respect to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

is given by

$$hx + ky + g(x + h) + f(y + k) + c = 0.$$

If the polar of P passes through Q the polar of Q passes through P, and if the polars of P and Q pass through R, then the polar of R is PQ.

The polar of P with respect to a circle centre O radius a is perpendicular to OP and, if it cuts OP in N, then ON . OP $= a^2$.

If $S_1 = 0$, $S_2 = 0$ represent two circles, where

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1,$$
$$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2,$$

then $S_1 + \lambda S_2 = 0$ represents any circle through the points of intersection (real, imaginary, or coincident) of the two circles, except when $\lambda = -1$, which gives the radical axis of the two circles.

The radical axis of two circles is perpendicular to the line of centres, and the three radical axes of three given circles taken in pairs are concurrent.

A co-axal system of circles (i.e. any pair of circles of the system has the same radical axis) is given by $x^2 + y^2 + 2gx + c = 0$, where c is a fixed constant and g an arbitrary constant.

The limiting points of the system are real if c is positive and imaginary if c is negative.

The circles $S_1 = 0$ and $S_2 = 0$ cut orthogonally (at right angles) if

$$2g_1g_2 + 2f_1f_2 = c_1 + c_2.$$

The Parabola

4. Canonical form of the equation of a parabola is $y^2 = 4ax$ and its latus rectum is $4a$.

Slope equation of tangent is $y = mx + a/m$.

Equation of tangent at (x_1, y_1) is

$$yy_1 = 2a(x + x_1).$$

Equation of normal at (x_1, y_1) is

$$y - y_1 = \frac{-y_1}{2a}(x - x_1).$$

Equation of normal of slope m is

$$y = mx - 2am - am^3.$$

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Subnormal is constant and equals $2a$.

The parametric equation of the parabola is $x = at^2$, $y = 2at$.

If the parameter at one end of a focal chord is t_1, the parameter of the other end is $-1/t_1$.

Tangents at ends of a focal chord intersect at right angles on the directrix.

The polar of (h, k) with respect to the parabola $y^2 = 4ax$ is

$$ky = 2a(x + h).$$

If the polar of P passes through Q, the polar of Q passes through P, and, if the polars of P and Q pass through R the polar of R is PQ.

The polar of the focus of a parabola with respect to the parabola is the directrix.

If P is a point on the parabola $y^2 = 4ax$, S its focus, T the point in which the tangent at P meets OX, PN the ordinate of P, A its vertex, PM the perpendicular from P on the directrix KL, G the point in which the normal at P meets OX, PQ a focal chord and PT cuts KL in R, then

$$TA = AN, \quad TS = SP,$$

PT bisects the angle SPM, $\angle PSR = 90°$, SM is perpendicular to PR, and SM meets PR on the tangent at A.

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EXAMPLES IX

1. Prove that the angle θ between the pair of straight lines

$$ax^2 + 2hxy + by^2 = 0$$

is given by

$$\tan\theta = 2(h^2 - ab)^{\frac{1}{2}}/(a + b).$$

One of the medians of the triangle formed by the straight lines $ax^2 + 2hxy - ay^2 = 0$ and the the line $px + qy = r$ lies along the y-axis. If a and r are both different from zero, prove that

$$ap + hq = 0.$$

2. Prove that, if $\mu \leqslant 169/56$, there are two finite real values of λ for which the equation

$$9x^2 + \lambda xy + \mu y^2 - 45x + 13y + 14 = 0$$

represents a pair of straight lines.

If $\mu = -1$, find the separate equations of the lines constituting *one* of these pairs.

3. Show that the bisectors of the angles between the line-pair $ax^2 + 2hxy + by^2 = 0$ are given by the equation

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}.$$

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Hence show that the line $px + qy + r = 0$ forms with the given line-pair an isosceles triangle if $h(p^2 - q^2) = (a - b)pq$.

Find also the conditions that the triangle should be equilateral.

4. Find the condition that the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

should represent a pair of straight lines.

Show that $x^2 + 4xy - 2y^2 + 6x - 12y - 15 = 0$ represents a pair of straight lines and that these lines together with the pair of lines $x^2 + 4xy - 2y^2 = 0$ form a rhombus.

5. Obtain a single equation which represents the bisectors of the angles between the lines $ax^2 + 2hxy + by^2 = 0$.

Find the condition for the lines $a_1x^2 + 2h_1xy + b_1y^2 = 0$ to have the same bisectors as the first given pair, and show that every such pair of lines can be represented by an equation of the form

$$ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0.$$

Find the equation of the pair of lines, one of which passes through the point (α, β), and whose bisectors are $x^2 - y^2 = 0$.

6. Find the condition that $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents a pair of straight lines.

Prove that $x^2 - y^2 + 2xy \sinh \theta + 2ax \cosh \theta + a^2 = 0$ represents a pair of straight lines for all values of θ, and show that the locus of their point of intersection is the circle $x^2 + y^2 = a^2$.

7. Find the equation of the pair of lines joining the origin to the points in which the pair of lines

$$4x^2 - 15xy - 4y^2 + 39x + 65y - 169 = 0$$

are met by the line $x + 2y - 5 = 0$.

Show that the quadrilateral having the first pair, and also the second pair, as adjacent sides, is cyclic, and find the equation of its circumcircle.

8. The lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ are turned in their own plane about the origin O of co-ordinates through an acute angle $\tan^{-1} 2$ in the sense from OX to OY. Prove that, in their new position, their equation referred to the original axes is

$$(a - 4h + 4b)x^2 + (4a - 6h - 4b)xy + (4a + 4h + b)y^2 = 0.$$

Prove that the length of the chord joining the points of intersection of the above pair of lines in either position with the circle $x^2 + y^2 = r^2$ is

$$r\left[2 - \frac{2(a + b)}{\sqrt{\{(a - b)^2 + 4h^2\}}}\right]^{\frac{1}{2}}.$$

9. Find the equation of the polar of the point (x_1, y_1) with respect to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

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Prove that the circles with respect to which the given straight line $lx + my + n = 0$ is the polar of the origin form a co-axial system with radical axis $2lx + 2my + n = 0$. Find the co-ordinates of the limiting points of this system and the equation of the system of circles which cuts the system orthogonally.

10. The circle $x^2 + y^2 = a^2$ intersects the x-axis at the points $A(a, o)$ and $B(-a, o)$ and P is the point $(a \cos \theta, a \sin \theta)$. Prove that the equations of the lines AP and BP are

$$(x - a) \cos (\theta/2) + y \sin (\theta/2) = 0,$$

and

$$(x + a) \sin (\theta/2) - y \cos (\theta/2) = 0.$$

The lines AB and CD are two perpendicular diameters of a circle, and P is a variable point on its circumference. The straight line AP meets CD in Q and the straight line through Q parallel to AB meets BP in R. Prove that the locus of the point R is a parabola passing through the points B, C, and D. What position of the point P corresponds to the point at infinity on the parabola?

11. Prove that the equation of the circle having the points (x_1, y_1) and (x_2, y_2) as extremities of a diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

The line $x \sin \alpha + y \cos \alpha = 1$ intersects the conic

$$ax^2 + 2hxy + by^2 = 1$$

at points P, Q, which lie at finite distances from the origin.

Find the equation of the circle on PQ as diameter and prove that, if this circle passes through the origin and has its centre on the x-axis, then

$$a + b = 1 \quad \text{and} \quad h = a \cot \alpha.$$

12. Show that the line $(x - a) \cos \theta + y \sin \theta = r$ touches the circle $(x - a)^2 + y^2 = r^2$, and find the co-ordinates of the point of contact.

A pair of parallel tangents is drawn to a given circle, and another pair, perpendicular to these, is drawn to a second given circle of equal radius.

Prove that each diagonal of the square formed by the four tangents passes through a fixed point.

13. A co-axial system of circles contains the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

and one of its limiting points is $(-g, o)$; find the equation of the radical axis and the co-ordinates of the other limiting point.

Show that the system of circles orthogonal to this system is represented by the equation

$$f(x^2 + y^2) + 2fgx - (g^2 - c)y + fg^2 + \mu(x + g) = 0,$$

where μ is a parameter.

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14. Prove that, for all values of the constants λ and μ, the circle whose equation is

$$(x - \alpha)(x - \alpha + \lambda) + (y - \beta)(y - \beta + \mu) = r^2$$

bisects the circumference of the circle

$$(x - \alpha)^2 + (y - \beta)^2 = r^2.$$

Find the equation of the circle which bisects the circumference of the circle $x^2 + y^2 + 2y - 3 = 0$ and touches the line $x - y = 0$ at the origin.

15. Define the radical axis of two circles and show that the family of circles given by the equation $x^2 + y^2 + 2\lambda x + c = 0$, where λ is a parameter, form a co-axial system.

OX and OY are two perpendicular lines. P is a variable point on OX, and the polar of P with respect to a given fixed circle in the plane OXY intersects OY at the point Q. Prove that, for different positions of P on OX, the circles on PQ as diameter form a co-axial system whose radical axis passes through the centre of the given fixed circle.

16. Define the radical axis of a pair of circles and find its equation for the circles

$$x^2 + y^2 + 2gx + 2fy + c = 0, \quad x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0.$$

Prove that the radical axis of three circles taken in pairs, are, in general, concurrent.

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Prove that the point of concurrence is the centre of a circle which cuts all three circles orthogonally.

Find the equation of the circle which is orthogonal to all the circles

$$x^2 + y^2 + 2x + 4y + 3 = 0, \quad x^2 + y^2 + 3x + 5y = 0,$$
$$x^2 + y^2 + 4x + 5y - 1 = 0.$$

17. Find the equation of the tangent to the parabola $y^2 = 4ax$ parallel to the line $ly - x = 0$ and the co-ordinates of the point of contact.

Prove that the equation of the locus of the point of intersection of tangents to the parabola $y^2 = 4ax$, which intercept a constant length c on the directrix is $(y^2 - 4ax)(x + a)^2 = c^2x^2$.

18. Find the equation of the tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

The curves p and q are arcs of parabolas. The equation of p is $y = +2\sqrt{ax}$ and that of q is $y = +2\sqrt{-ax}$, a being a positive constant. The tangent to p at the point $(at^2, 2at)$ intersects q at Q, and the tangent to q at Q intersects p at the point $(at_1^2, 2at_1)$. Prove that

$$t_1 = (\sqrt{2} - 1)^2 t.$$

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19. Prove that the tangent to the parabola $y^2 = 4kx$ at the point $(kt^2, 2kt)$ is the line $ty = x + kt^2$.

A tangent to the parabola $y^2 + n^2ax = 0$, where n is positive, intersects the parabola $y^2 = 4ax$ at P and Q. Prove that the locus of the mid-point of PQ is another parabola and that, if the latus rectum of this parabola is $16a/17$, then $n = 3$.

20. Obtain the equation of the normal at the point $(at^2, 2at)$ on the parabola $y^2 = 4ax$.

Prove that, in general, three normals can be drawn to the parabola $y^2 = 4ax$ to pass through a given point P. If two of these three normals are perpendicular, prove that P lies on the parabola $y^2 = a(x - 3a)$.

21. Obtain the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

Two triangles are formed respectively by the tangents and normals to the parabola at any three of its points; prove that the line joining their orthocentres is parallel to the axis of the parabola.

22. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

Prove that the three normals to the parabola pass through a general point (X, Y) and that the parameters of the feet of the normals satisfy the relation $t_1 + t_2 + t_3 = 0$.

Prove also that the equation of the circle through the feet of the normals is

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$$x^2 + y^2 - (X + 2a)x - \tfrac{1}{2}Yy = 0.$$

23. Prove that the equation $2yt = x + (t^2 - 1)a$ . . . (1), where a and t are parameters, represents a family of parallel straight lines when t is fixed and a is variable, and write down the equation of that straight line which passes through the origin.

When a is fixed and t is variable, prove that the equation (1) represents a family of tangents to the parabola $y^2 = a(x - a)$.

When a and t both vary, find the relation which must hold between a and t in order that the equation (1) shall represent the family of tangents to the circle $x^2 + y^2 = 1$.

24. Find the equation of the tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

Find the co-ordinates of the centre of the circle which touches the parabola at this point and passes through the focus, and show that, as t varies, the locus of the centres of these circles is the curve $27ay^2 = (2x - a)(5a - x)^2$.

25. Find the angle between the two lines whose combined equation is $ax^2 + 2hxy + by^2 = 0$, and show that the lines

$$(m + 1)y^2 - (m^2 + 2m - 1)xy + (m^2 - m)x^2 = 0$$

intersect each other at an angle $\pi/4$ radians for all values of m.

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Find the equations of the chords of the parabola $y^2 = 4ax$ that pass through the point $(-6a, 0)$ and which subtend an angle of $\pi/4$ radians at the vertex.

26. Find the equation of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

If P be the point $(at^2, 2at)$ and the normal at P meets the parabola again at the point Q, whose co-ordinates are $(at_1^2, 2at_1)$, show that

$$t^2 + tt_1 + 2 = 0.$$

If M be the mid-point of PQ and N is the mid-point of PM, show that, as P varies on the parabola, the locus of N is the parabola whose equation is $y^2 = a(x - 3a)$.

27. The chord PQ, joining the points P $(at_1^2, 2at_1)$, Q $(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$, passes through the focus.

Find the relation between t_1 and t_2.

Prove that the circle on PQ as diameter touches the directrix of the parabola, and that its point of contact is the point of intersection of the tangents to the parabola at P and Q.

28. PQ is a focal chord of a parabola with focus S, and the tangents at P, Q meet at R. Show that

(i) the line through R parallel to the axis of the parabola bisects PQ ;

(ii) if the line through R and the mid-point M of PQ cuts the parabola at K, then RK = KM ;

(iii) the circles drawn through S to touch the parabola at P, Q respectively are orthogonal.

29. Tangents from two points P and Q on a parabola meet in T. Prove that (i) PT and QT subtend equal angles at the focus S, (ii) ST makes with either tangent the same angle as the axis of the parabola makes with the other tangent.

Deduce that the circle circumscribing a triangle formed by three tangents to the parabola passes through the focus.

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CHAPTER X

The Ellipse and Hyperbola

The Ellipse

1. The canonical form of the equation of an ellipse is

$$x^2/a^2 + y^2/b^2 = 1,$$

where $b^2 = a^2(1 - e^2)$, and e is the eccentricity ($e < 1$).

The latus rectum is $2b^2/a$.

The slope equation of the tangent to ellipse $x^2/a^2 + y^2/b^2 = 1$ is

$$y = mx \pm \sqrt{(a^2m^2 + b^2)}.$$

The tangent at (x_1, y_1) has the equation

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

The normal at (x_1, y_1) is $\dfrac{x - x_1}{x_1/a^2} = \dfrac{y - y_1}{y_1/b^2}$.

The line $lx + my + n = 0$ touches the ellipse if

$$a^2l^2 + b^2m^2 = n^2.$$

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The polar of (h, k) with respect to the ellipse $x^2/a^2 + y^2/b^2 = 1$ is

$$hx/a^2 + ky/b^2 = 1.$$

If the polar of P passes through Q the polar of Q passes through P, and if the polars of P and Q pass through R, the polar of R is PQ.

The Director Circle of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is the locus of the points of intersection of perpendicular tangents and has the equation $x^2 + y^2 = a^2 + b^2$.

The auxiliary circle has as its equation $x^2 + y^2 = a^2$.

If ϕ be the eccentric angle of the point P on the ellipse, then

$$\mathrm{P} \equiv (a \cos \phi,\ b \sin \phi).$$

The equation of the chord joining points on the ellipse whose eccentric angles are ϕ_1 and ϕ_2 is given by

$$\frac{x}{a} \cos \tfrac{1}{2}(\phi_1 + \phi_2) + \frac{y}{b} \sin \tfrac{1}{2}(\phi_1 + \phi_2) = \cos \tfrac{1}{2}(\phi_1 - \phi_2).$$

The equation of the tangent at $(a \cos \phi,\ b \sin \phi)$ is

$$\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 1.$$

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The equation of the normal at $(a \cos \phi,\ b \sin \phi)$ is

$$\frac{ax}{\cos \phi} - \frac{by}{\sin \phi} = a^2 - b^2.$$

Two diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$ are conjugate (i.e. each bisects all chords parallel to the other), if the product of their slopes is $-b^2/a^2$.

The eccentric angles of the extremities of two conjugate diameters differ by $\pi/2$.

If C be the centre of the ellipse and P and V two extremities of two conjugate diameters $CP^2 + CV^2 = a^2 + b^2$.

The equations of equi-conjugate diameters are given by

$$y = \pm \frac{b}{a} x.$$

C is the centre of the ellipse $x^2/a^2 + y^2/b^2 = 1$ and P any point on it; PT is the tangent at P meeting CX in T and CY in t; SR and S′R′ are the perpendiculars from the foci S and S′ on the tangent at P; SR is produced to meet S′P in D; PG is the normal at P meeting CX in G and CY in g; CF is a line parallel to PT meeting Pg in F and the ellipse at V.

Then $\quad CN \,.\, CT = a^2;\ PN \,.\, Ct = b^2;$

$CG = e^2 x_1$ and PG bisects $\angle$ SPS′; $P \equiv (x_1, y_1)$; $PF \,.\, PG = b^2$; $SR \,.\, S'R' = b^2$; the locus of R and R′ is the auxiliary circle.

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$$SP + S'P = 2a.$$

The Hyperbola

2. The canonical equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where $b^2 = a^2(e^2 - 1)$, and e is its eccentricity $(e > 1)$.

The latus rectum $= 2b^2/a$.

If S and S′ are the two foci and P any point on the hyperbola

$$S'P - SP = 2a.$$

The equations of tangents of slope m to the hyperbola

$$x^2/a^2 - y^2/b^2 = 1$$

are given by

$$y = mx \pm \sqrt{(a^2m^2 - b^2)}.$$

The equation of the tangent at (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

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The equation of the normal at (x_1, y_1) is

$$\frac{x - x_1}{x_1/a^2} + \frac{y - y_1}{y_1/b^2} = 0.$$

The line $lx + my + n = 0$ touches the hyperbola if

$$a^2l^2 - b^2m^2 = n^2.$$

The equation of the polar of (h, k) with respect to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ is $hx/a^2 - ky/b^2 = 1$.

If the polar of P passes through Q, then the polar of Q passes through P, and, if the polars of P and Q pass through R the polar of R is PQ.

The equation of the director circle of the hyperbola

$$x^2/a^2 - y^2/b^2 = 1$$

(i.e. the locus of the points of intersection of perpendicular tangents) is

$$x^2 + y^2 = a^2 - b^2.$$

The auxiliary circle is $x^2 + y^2 = a^2$.

The parametric equation of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is

$$x = a \sec \phi, \quad y = b \tan \phi.$$

The lines $y = m_1x$ and $y = m_2x$ are conjugate diameters of the hyperbola if $m_1m_2 = b^2/a^2$.

Using the same notation as for the ellipse

$$\mathrm{CN} \,.\, \mathrm{CT} = a^2; \quad \mathrm{CG} = e^2x_1; \quad \mathrm{SR} \,.\, \mathrm{S'R'} = b^2.$$

The locus of R and R′ is the auxiliary circle.

The normal at P bisects the angle between the focal radii at P and the tangent at P bisects the external angle between the focal radii.

The asymptotes of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ are given by

$$x^2/a^2 - y^2/b^2 = 0.$$

If 2α be the angle between the asymptotes, then the eccentricity $e = \sec \alpha$.

The equation of the rectangular hyperbola $x^2 - y^2 = a^2$ referred to its asymptotes as axes is $xy = c^2$.

The tangent at (x_1, y_1) to this hyperbola is $x\,y_1 + y\,x_1 = 2c^2$.

The parametric equation of the hyperbola $xy = c^2$ is

$$x = ct, \quad y = c/t.$$

EXAMPLES X

1. Find the equation of the chord of the ellipse $x^2/a^2 + y^2/b^2 = 1$ which is bisected at the point (α, β).

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Chords of the ellipse $x^2/a^2 + y^2/b^2 = 1$ are drawn to pass through a fixed point Q. Prove that the mid-points of the chords lie on an ellipse whose centre is at the mid-point of the line joining Q to the centre of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

2. Prove that the eccentric angles of the ends of two conjugate diameters of an ellipse differ by a right angle.

The centre of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is C and CP, CQ are a pair of conjugate semi-diameters. If the eccentric angle at P is θ and the chord PQ passes through the point $(a\sqrt{2}, 0)$, prove that $\theta = \pm\frac{\pi}{12}, \pm\frac{7\pi}{12}$.

3. Any point P is taken on an ellipse whose foci are A, B. Q is a point on the bisector of the angle APB and L is the foot of the perpendicular from P on to AB, produced if necessary.

From Q the perpendicular QM is drawn to PA. If PL = QM prove that the length of PM has the constant value b^2/ae, where a, b are the principal semi-axes and e is the eccentricity.

4. Assuming the focus-directrix definition of an ellipse, show that its equation can be put in the form $x^2/a^2 + y^2/b^2 = 1$.

Tangents are drawn to an ellipse of eccentricity e from a point on a concentric circle. Prove that the chord of contact of the tangents touches a concentric co-axial ellipse of eccentricity $e\sqrt{2 - e^2}$.

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5. Prove that the line $lx + my + n = 0$ touches the ellipse

$$x^2/a^2 + y^2/b^2 = 1 \text{ if } a^2l^2 + b^2m^2 = n^2.$$

Lines are drawn through the origin perpendicular to the tangents from a point P to the above ellipse.

If the lines are conjugate diameters of the ellipse, prove that P lies on the curve $a^2x^2 + b^2y^2 = a^4 + b^4$.

6. Find the equation of the tangent to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ at the point (x_1, y_1) and deduce that the equation of the chord of contact of the tangents from (x_2, y_2) to the ellipse is

$$b^2xx_2 + a^2yy_2 = a^2b^2.$$

A point P moves so that the chord of contact of the tangents from P to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ touches the ellipse $4(b^2x^2 + a^2y^2) = a^2b^2$. Find the locus of P.

7. (i) Define an ellipse, and show from your definition that its equation can be put in the form $x^2/a^2 + y^2/b^2 = 1$.

(ii) P, Q are variable points of the ellipse such that the lines joining them to its centre are perpendicular and the tangents at P and Q meet at T. Find the locus of T.

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8. A chord PQ of an ellipse intersects the directrix corresponding to the focus S at R. Show that SR is the external bisector of angle PSQ.

S and H are the two real foci of an ellipse and the tangent at P on the ellipse cuts the directrix corresponding to S at T and the directrix corresponding to H at K. Show that the triangles STP and HKP are similar.

9. U is a point inside an ellipse with foci S, S′; SU, S′U meet the ellipse at points P′, P respectively on the same side of SS′ as U; and the tangents at P, P′ meet at T. By considering the triangles SPT, SPU, or otherwise, prove that the angle SUP is twice angle STP.

Prove that the angle STP is equal to angle S′TP′.

Let SP meet S′P′ at V. By considering the triangles S′PT, S′PV, or otherwise, prove that the supplement of angle S′VP is twice angle S′TP. Hence show that S, U, S′, V are concyclic if TP, TP′ are perpendicular.

10. Prove that the focal distances from any point P on an ellipse are equally inclined to the tangent at that point.

The foci of an ellipse are S, S′ and P is any point on the curve. If the normal at P intersects the line SS′ at G, prove that

$$PG^2 = SP \cdot S'P(1 - e^2),$$

where e is the eccentricity of the ellipse.

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11. (*a*) Prove that the locus of all points P from which the tangents to a fixed ellipse are perpendicular to each other is a circle concentric with the ellipse.

(*b*) Two tangents to the parabola $y^2 = 4ax$ meet at right angles at P. Their points of contact are A, B. Prove that the gradient of PA is numerically equal to PB/PA.

12. (i) Prove that the lines which join the ends of a diameter of an ellipse to any point on the ellipse are parallel to conjugate diameters.

(ii) A parallelogram circumscribes an ellipse. Prove that its diagonals lie along a pair of conjugate diameters of the ellipse.

13. Prove that in a hyperbola, that part of the tangent intercepted by the asymptotes is bisected at the point of contact.

The tangent at a point P of a rectangular hyperbola whose centre is C meets the asymptotes in H and K, and the normal at P meets its real axis in G. Prove that C, G, H, K lie on a circle centre P.

14. Prove that the point P whose co-ordinates are $x = a \sec \theta, y = a \tan \theta$, lies on the rectangular hyperbola $x^2 - y^2 = a^2$.

If Q be the point whose parameter is $(\theta + \pi/2)$ and R (x_1, y_1) is the mid-point of PQ, prove that $y_1/x_1 = \sin \theta + \cos \theta$ and find the locus of R.

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15. Find the equation of the chord joining the two points $(ct_1, c/t_1)$ and $(ct_2, c/t_2)$ on the rectangular hyperbola $xy = c^2$, and deduce that the equation of the tangent to the curve at the point $(ct, c/t)$ is

$$t^2y + x = 2ct.$$

The points P $(ct_1, c/t_1)$, Q $(ct_2, c/t_2)$ and R $(ct_3, c/t_3)$ on the hyperbola form a triangle right angled at Q; prove that $t_1t_2^2t_3 = -1$ and deduce that the tangent to the hyperbola at Q is perpendicular to the straight line PR.

16. The co-ordinates of a point P are given parametrically in terms of a real variable t by the relations $x = a + b\cosh t + c\sinh t$, $y = p + q\cosh t + r\sinh t$. Show that in general, the locus of P is one branch of a hyperbola.

Prove that $q^2 - r^2 = c^2 - b^2$ is the condition that the hyperbola should be rectangular.

17. Prove that the straight line $lx + my + n = 0$ touches the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ if $a^2l^2 - b^2m^2 = n^2$.

The circle $x^2 + y^2 - 2\alpha y + \beta = 0$ cuts one asymptote of the hyperbola in the distinct points P, P′; and the other in Q, Q′; where P and Q are on the same side of the y-axis. If PQ is a tangent to the hyperbola, prove that the circle must pass through the foci of the hyperbola.

18. A rectangular hyperbola is cut by any circle in four points. Prove that the sum of the squares of the distances of these four points from the centre of the hyperbola is equal to the square on the diameter of the circle.

19. Find the equation of the normal to the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ at the point (x_1, y_1) on the curve.

If A is a fixed point (h, k) not lying on the hyperbola, prove that, in general, four normals can be drawn from A to the hyperbola, and that their feet lie on the curve

$$(a^2 + b^2)xy - b^2kx - a^2hy = 0.$$

Prove that this curve is a rectangular hyperbola and find its asymptotes.

20. A circle of variable radius r moves so as always to touch each of the two fixed circles $(x - 1)^2 + y^2 = 4$; $(x + 1)^2 + y^2 = 9$.

If both fixed circles lie inside the variable one, prove that its centre lies on the curve whose equation is $12x^2 - 4y^2 = 3$.

21. Find the equation of the chord of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ whose mid-point is the general point (α, β) and interpret your result in the special case when the point (α, β) lies on the hyperbola.

A variable chord of the hyperbola is a tangent to the circle $x^2 + y^2 = c^2$. Prove that the locus of the mid-point of the chord is the curve

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 = c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right).$$

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22. The tangent at a point P on a hyperbola k meets the asymptotes at Q and R ; prove that P is the mid-point of QR.

C is the centre of k, and the normal at P meets the axes of k at A and B ; prove that C, A, B, Q, R are concyclic.

23. A chord AB of one branch of a hyperbola cuts the asymptotes at C and D ; prove that AD = CB.

A variable line passes through a fixed point A and meets two fixed intersecting lines at B and C on opposite sides of A. Find the locus of the middle point of BC.

24. R and r ($R > r$) are the radii of two fixed coplanar circles, centres A and B, external to each other. Show that the locus of the centres of circles which touch these two circles consists of branches of a hyperbola.

Distinguish between the cases when the fixed circles touch the variable circle (i) externally, (ii) one internally and the other externally, and (iii) internally. Find the asymptotes in each case, and, if $AB = (R - r)\sqrt{2}$, show that one of the hyperbolas is rectangular.

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CHAPTER XI

Polar and General Equation of a Conic

1. Polar Equation of a Conic

The polar equation of a conic of semi-latus rectum l, eccentricity e is

$$\frac{l}{r} = 1 + e \cos \theta.$$

The equation of its directrix is

$$\frac{l}{r} = e \cos \theta.$$

The equation of the chord joining the points whose vectorial angles are $(\alpha + \beta)$ and $(\alpha - \beta)$ is

$$\frac{l}{r} = e \cos \theta + \sec \beta \cos (\theta - \alpha).$$

Equation of the tangent at the point (r, α) is

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$$\frac{l}{r} = e \cos \theta + \cos (\theta - \alpha).$$

2. The General Equation of a Conic (cartesian co-ordinates) is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

This represents

(*a*) Two straight lines if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.
(*b*) A circle if $a = b$ and $h = 0$.
(*c*) A parabola if $h^2 = ab$.
(*d*) An ellipse if $h^2 < ab$.
(*e*) A hyperbola if $h^2 > ab$.

The equation of the asymptotes of the above general conic (only real in the case of the hyperbola) are given by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c - \frac{\Delta}{ab - h^2} = 0,$$

where $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

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The centre of the general conic is

$$\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right).$$

The principal semi-axes of the central conic

$$ax^2 + 2hxy + by^2 = 1$$

are of length r given by the equation

$$h^2 = \left(a - \frac{1}{r^2}\right)\left(b - \frac{1}{r^2}\right),$$

and their equations are given by

$$hx + \left(b - \frac{1}{r^2}\right)y = 0,$$

where the values of r^2 are obtained from the previous equation.

EXAMPLES XI

1. Show that the equation $(x/a)^{\frac{1}{2}} + (y/b)^{\frac{1}{2}} = 1$ represents a parabola touching the co-ordinate axes, and that the line

$$y = mx + abm/(am - b)$$

is a tangent to the curve for all values of m.

Hence, or otherwise, prove that the equation of the directrix is $ax + by = 0$, and obtain the co-ordinates of the focus.

2. Prove that the general equation of the second degree represents a parabola if the second degree terms form a perfect square.

Prove that the latus rectum of the parabola

$$9x^2 + 6xy + y^2 + 2x + 3y + 4 = 0$$

is $7\sqrt{10}/100$, and sketch the curve.

3. Prove that the centre of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

lies at the point $\left(\frac{fh - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$.

Find the centre of the conic

$$13x^2 - 18xy + 37y^2 + 2x + 14y - 2 = 0,$$

and show that the curve is an ellipse of semi axes $(\sqrt{(2/5)}$ and $\frac{1}{2}\sqrt{(2/5)}$.

4. The straight line $(x - \alpha)/\cos\phi = (y - \beta)/\sin\phi$, passing through the point P (α, β) intersects at the points Q and R the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

where a, b, c, f, g, h are real.

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If the algebraic lengths of the segments PQ and PR are r_1 and r_2, prove that r_1 and r_2 are the roots of the quadratic equation

$$r^2(a\cos^2\phi + 2h\cos\phi\sin\phi + b\sin^2\phi) + 2r\{(a\alpha + h\beta + g)\cos\phi + (h\alpha + b\beta + f)\sin\phi)\} + a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0.$$

Deduce that, if $ab > h^2$ the conic is an ellipse. Also find the co-ordinates of the centre of the conic.

5. If an ellipse is represented by the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \text{ prove that}$$

(i) the product ab is necessarily positive ;

(ii) the origin is outside or inside the ellipse according as the product ac is positive or negative.

A diameter of the conic $2x^2 + xy + 3y^2 = 6x + 7y + 44$ is parallel to the line $y = 2x$. Find where this diameter, produced if necessary, meets the y-axis.

6. Prove that the lengths of the semi-axes of the conic

$$ax^2 + 2hxy + by^2 = c$$

are given by the roots of the equation

$$(ab - h^2)r^4 - (a + b)cr^2 + c^2 = 0.$$

If a, b, c are positive and $ab < h^2$ show that the curve is a hyperbola.

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Find the eccentricity of the conic

$$2x^2 + 2xy\sqrt{18} + 5y^2 = 1.$$

7. Find the general equation of a conic which passes through the origin O and has the axes Ox, Oy as tangent and normal at O.

If O is a fixed point of a conic s, and P, Q are variable points of s, such that the chord PQ subtends a right angle at O, prove that, in general, PQ meets the normal at O in a fixed point.

Show that, in the particular case in which s is a rectangular hyperbola, the chords PQ are all parallel to the normal at O.

8. If Ox, Oy, and Ox', Oy' are two pairs of rectangular axes through the same origin, and the second pair is obtained from the first by a positive rotation through an angle θ, obtain expressions for x and y in terms of x', y'.

If the equation of a central conic S, referred to rectangular axes Ox, Oy through its centre is $7x^2 + 12xy + 2y^2 = 1$, determine the angle θ so that the new axes Ox', Oy' are the principal axes of S.

By transforming the equation of S, find whether S is an ellipse or a hyperbola and obtain the lengths of its semi-axes.

9. Prove that the locus of the foot of the perpendicular from a focus to a tangent to a central conic is a circle.

What is the corresponding result for a parabola ?

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Given the focus S and the two tangents t, t' to a conic, show that the locus of the centre of the conic is a straight line and that the minor axis is a tangent to a fixed parabola which has S as a focus.

10. Show that the polar equation of a conic section is $l/r = 1 + e \cos \theta$, the pole being at the focus and the initial line being perpendicular to the directrices.

An ellipse of semi-latus rectum l, eccentricity e and foci S, S′ is intersected at P and Q by a parabola of semi-latus rectum el, whose axis lies along SS′ and whose focus is at S.

The vertex of the parabola lies on the same side of S as does S′. Prove that the area of the triangle PSQ is

$$l^2 \frac{(1-e)}{(1+e)^{\frac{3}{2}}} \{e(2-e+e^2)\}^{\frac{1}{2}}.$$

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CHAPTER XII

Solid Co-ordinate Geometry—The Plane and Straight Line

1. If $(\bar{x}, \bar{y}, \bar{z})$ divide the join of (x_1, y_1, z_1), (x_2, y_2, z_2) in the ratio of $\lambda_2 : \lambda_1$, then

$$\bar{x} = \frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2}, \quad \bar{y} = \frac{\lambda_1 y_1 + \lambda_2 y_2}{\lambda_1 + \lambda_2}, \quad \bar{z} = \frac{\lambda_1 z_1 + \lambda_2 z_2}{\lambda_1 + \lambda_2}.$$

The length of the line joining (x_1, y_1, z_1), (x_2, y_2, z_2) is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

If a straight line make angles α, β, γ with OX, OY, OZ respectively, then the direction cosines of the line are

$$l = \cos\alpha, \quad m = \cos\beta, \quad n = \cos\gamma,$$

and

$$l^2 + m^2 + n^2 = 1,$$

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2.$$

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If the direction cosines of the line are proportional to f, g, h respectively, the actual direction cosines are

$$\frac{f, g, h,}{\sqrt{(f^2 + g^2 + h^2)}}.$$

Equations of a Plane

2. A plane making intercepts a, b, c on OX, OY, OZ respectively has the equation

$$x/a + y/b + z/c = 1.$$

If the perpendicular on a plane from O is of length p, and its direction cosines (actual) are l, m, n, its equation is (p always positive)

$$lx + my + nz = p.$$

The general equation of a plane is

$$ax + by + cz = d,$$

and the perpendicular on this plane from O is

$$\pm \frac{d}{\sqrt{(a^2 + b^2 + c^2)}}.$$

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The plane through (x_1, y_1, z_1) whose normal has the direction cosines (l, m, n) is

$$l(x - x_1) + m(y - y_1) + n(z - z_1) = 0$$

(direction cosines can be proportional ones).

The plane through the three points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

The Straight Line

3. The equation of the line through (x_1, y_1, z_1) having direction cosines (l, m, n) is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}.$$

Equation of the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}.$$

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The simultaneous equations

$$a_1x + b_1y + c_1z + d_1 = 0, \quad a_2x + b_2y + c_2z + d_2 = 0.$$

represent the line of intersection of the two planes represented by the separate equations.

Angles between Lines and Planes

4. The angle θ between lines of actual direction cosines (l_1, m_1, n_1), (l_2, m_2, n_2) is given by

$$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2,$$

and the lines are perpendicular if

$$l_1l_2 + m_1m_2 + n_1n_2 = 0.$$

The angle θ between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{\{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)\}}},$$

and the planes are perpendicular if

$$a_1a_2 + b_1b_2 + c_1c_2 = 0.$$

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Perpendiculars, etc.

5. The length of the perpendicular from (x_1, y_1, z_1) on the plane $lx + my + nz = p$ is

$$p - lx_1 - my_1 - nz_1.$$

The length of the perpendicular from (x_1, y_1, z_1) on the plane $ax + by + cz + d = 0$ is

$$\pm \frac{(ax_1 + by_1 + cz_1 + d)}{\sqrt{a^2 + b^2 + c^2}},$$

where the + sign is taken for d positive and the − sign for d negative.

The equation of the planes bisecting the angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$ is

$$\frac{a_1x + b_1y + c_1z + d}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}} = \pm \frac{(a_2x + b_2y + c_2z + d_2)}{\sqrt{(a_2^2 + b_2^2 + c_2^2)}}.$$

The length (p) of the perpendicular from (x_1, y_1, z_1) on the line $\frac{x - x_2}{l} = \frac{y - y_2}{m} = \frac{z - z_2}{n}$ is given by

$$p^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - \left[\frac{l(x_1 - x_2) + m(y_1 - y_2) + n(z_1 - z_2)}{\sqrt{(l^2 + m^2 + n^2)}}\right]^2.$$

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6. The condition that the lines

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}, \quad \frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2},$$

shall intersect is

$$\begin{vmatrix} x_1 - x_2 & l_1 & l_2 \\ y_1 - y_2 & m_1 & m_2 \\ z_1 - z_2 & n_1 & n_2 \end{vmatrix} = 0,$$

and the equation of the plane in which these lines lie is

$$\begin{vmatrix} x - x_2 & l_1 & l_2 \\ y - y_2 & m_1 & m_2 \\ z - z_2 & n_1 & n_2 \end{vmatrix} = 0.$$

The equation of the common perpendicular of the lines

$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$, $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$ is given by

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ m_1n_2 - m_2n_1 & l_2n_1 - l_1n_2 & l_1m_2 - l_2m_1 \end{vmatrix} = 0,$$

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$$\cosh A + \cosh B = 2 \cosh \frac{A+B}{2} \cosh \frac{A-B}{2};$$

$$\cosh A - \cosh B = 2 \sinh \frac{A+B}{2} \sinh \frac{A-B}{2}.$$

$$2 \sinh x \cosh y = \sinh (x+y) + \sinh (x-y);$$

$$2 \cosh x \sinh y = \sinh (x+y) - \sinh (x-y).$$

$$2 \cosh x \cosh y = \cosh (x+y) + \cosh (x-y);$$

$$2 \sinh x \sinh y = \cosh (x+y) - \cosh (x-y).$$

$$\sinh^{-1} x = \log_e \{x + \sqrt{(x^2+1)}\};$$

$$\cosh^{-1} x = \log_e \{x \pm \sqrt{(x^2-1)}\} \quad (x > 1);$$

$$\tanh^{-1} x = \tfrac{1}{2} \log_e \frac{1+x}{1-x} \qquad (x^2 < 1).$$

6. Complex Numbers

$i = \sqrt{(-1)}$ and, if a and b are real $a + ib$ is a complex number whose modulus $r = \sqrt{(a^2+b^2)}$ and whose argument θ is given by

$$\sin \theta = \frac{b}{\sqrt{(a^2+b^2)}}, \quad \cos \theta = \frac{a}{\sqrt{(a^2+b^2)}}.$$

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De Moivre's Theorem: $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$, where n is rational.

$$e^{i\theta} = \cos \theta + i \sin \theta; \quad e^{-i\theta} = \cos \theta - i \sin \theta;$$

$$\cos \theta = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots \to \infty;$$

$$\sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots \to \infty.$$

$$\cos i\theta = \cosh \theta; \quad \sin i\theta = i \sinh \theta; \quad \tan i\theta = i \tanh \theta.$$

$$\cosh i\theta = \cos \theta; \quad \sinh i\theta = i \sin \theta; \quad \tanh i\theta = i \tan \theta.$$

nth roots of $a + ib = r(\cos \theta + i \sin \theta)$, where r is the modulus and θ the argument (amplitude), are given by

$$r^{1/n}\left[\cos \frac{2k\pi + \theta}{n} + i \sin \frac{2k\pi + \theta}{n}\right], \text{ where } k = 0, 1, 2, 3, \ldots, (n-1).$$

$a - ib$ is the conjugate complex number to $a + ib$, and vice versa.

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and
$$\begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_2 & m_2 & n_2 \\ m_1n_2 - m_2n_1 & n_1l_2 - n_2l_1 & l_1m_2 - l_2m_1 \end{vmatrix} = 0,$$
and the length of this common perpendicular is
$$\frac{\pm \begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{\{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2\}}}.$$

7. If $S_1 \equiv a_1x + b_1y + c_1z + d_1$, $S_2 \equiv a_2x + b_2y + c_2z + d_2$, then $S_1 + \lambda S_2 = 0$ represents any plane through the line of intersection of the planes $S_1 = 0$ and $S_2 = 0$, where λ is any constant.

The conditions that the planes $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$, $a_3x + b_3y + c_3z + d_3 = 0$ shall have a common line of intersection are
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0, \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0,$$
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0, \quad \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0.$$

8. The volume of the tetrahedron joining the points $(x_1\ y_1\ z_1)$, (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) is
$$\frac{1}{6} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

Any two non-intersecting lines can be represented by
$$y = mx,\ z = c\ ;\ y = -mx,\ z = -c.$$

EXAMPLES XII

1. Prove that any plane through the line L common to the two non-parallel planes $ax + by + cz + d = 0$, $a_1x + b_1y + c_1z + d_1 = 0$ has the equation
$$\lambda(ax + by + cz + d) + \mu(a_1x + b_1y + c_1z + d_1) = 0,$$
where λ, μ are finite constants and not both zero.

Find the equation of the plane through the line L which is parallel to the line $x/l = y/m = z/n$, and, hence or otherwise, show that the two lines are coplanar if, and only if,
$$d(la_1 + mb_1 + nc_1) = d_1(la + mb + nc).$$

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2. If $u = 0$, $v = 0$ are the equations of two planes, examine the nature of the locus defined by $\lambda u + \mu v = 0$, where λ and μ are constants. Find a and b so that the planes $x + y - z = 0$, $3x + by - 2z = 1$, $x - 3y + az = 2$, shall pass through one line, and determine the symmetrical form of the equation of this line. *(II)*

3. Find the equation of a plane whose perpendicular distance from the origin is p and whose normals have direction cosines l, m, n.

   Obtain the equation of the plane π (A, B) which bisects at right angles the straight segment AB, where A and B are the points (a, b, c), (d, e, f). Show that for any three points A, B, C the planes π (A, B), π (B, C), π (C, A) have, in general, a straight line, λ, in common.

   Show that, if A, B, C are the points (1, 0, 1), (0, 1, 1), (2, 2, 0), the distance from the origin is $(3\sqrt{22})/11$.

4. Find the equation of the plane through the line

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$$

parallel to the line

$$\frac{x-a_1}{l_1} = \frac{y-b_1}{m_1} = \frac{z-c_1}{n_1}.$$

   Prove that the lines

$$\frac{x-3}{1} = \frac{y-1}{2} = \frac{z+1}{3} \text{ and } \frac{x-2}{1} = \frac{y-5}{3} = \frac{z}{1}$$

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   intersect and find the co-ordinates of their common point and equation of their common plane.

5. The feet of the perpendiculars from P (x_1, y_1, z_1) to the line $x = a$, $y = z$ and to the plane $x + y + z = 3a$ are Q and R respectively; find the co-ordinates of Q and R.

   If QR is parallel to the plane $3x - y + z = 0$, prove that P must lie on the plane $x - y = 0$. Also find the locus of P when PQ = PR. *(II)*

6. Find the angle between the two lines whose direction cosines are (l, m, n, l_1, m_1, n_1).

   Find the equations of the projection of the line $2x = 2y = z$ on the plane $x - 2y - z + 6 = 0$. If the line and its projection on this plane lie in a vertical plane and if there is a horizontal line whose direction cosines are $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$, find the direction cosines of the vertical.

7. Obtain an expression for the angle between the lines

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}; \quad \frac{x-a_1}{l_1} = \frac{y-b_1}{m_1} = \frac{z-c_1}{n_1}.$$

   If A, B, C, D are four points such that $AB^2 + CD^2 = AC^2 + BD^2$, prove that BC is perpendicular to AD.

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8. Find the angle between the lines

$$\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1} \text{ and } \frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2}.$$

If A, B, C, D are four points in space such that AB is perpendicular to CD and AC is perpendicular to BD, prove that AD is perpendicular to BC.

9. Find the co-ordinates of the point P on the intersection of the two planes $2x + 2y - z = 0$, $x + y - 3z = 0$, which has a minimum distance from the straight line L passing through the points $(-2, 3, -2)$, $(-5, 5, -3)$.

Find the co-ordinates of the point Q on L which is nearest to P.

10. A fixed point A is on the axis of x and a fixed line OB lies in the plane $x = 0$ through the origin O. A variable line AP is drawn through A in the plane $z = 0$ and the shortest distance between AP and OB meets the former at P.

Prove that the locus of P is an ellipse which has OA for its minor axis. *(II)*

11. Find the length of the shortest distance between the two lines whose equations are

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \text{ and } \frac{x-a_1}{l_1} = \frac{y-b_1}{m_1} = \frac{z-c_1}{n_1}.$$

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Find the length and the equations of the common perpendicular to the lines $x = y = -z$, and $x + y = 0$, $z = a$.

12. Find the length and direction cosines of the shortest straight line joining the two straight lines whose equations are

$$\frac{x-6}{3} = \frac{y-7}{-1} = \frac{z-4}{1}; \quad \frac{x-3}{-3} = \frac{y+11}{2} = \frac{z+2}{4}. \qquad (II)$$

13. Find the length and equations of the shortest distance between the lines

$$\frac{x-6}{2} = \frac{y+4}{5} = \frac{z-2}{1}, \quad \frac{x+1}{-4} = \frac{y-9}{5} = \frac{z-5}{7}.$$

14. Show that the equations of two non-intersecting lines p, p_1 may be simultaneously expressed in the forms $y = mx$, $z = c$ and $y = -mx$, $z = -c$.

The common perpendicular to p, p_1 meets them at A, A_1 respectively. Points P, P_1 are taken on p, p_1 respectively so that AP . A_1P_1 is constant. Prove that the locus of the mid-point of the segment PP_1 is a pair of conjugate hyperbolas whose common asymptotes are parallel to p, p_1.

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15. Prove that the equations of any two straight lines may be expressed as

$$y = kx,\ z = c\ ;\quad y = -kx,\ z = -c.$$

If a variable line intersects both these lines and is equally inclined to them, prove that it lies on one or other of the surfaces

$$yz = kcx,\quad kzx = cy.$$

16. Prove that the perpendicular distance of the plane $ax + by + cz = d$ from the origin of co-ordinates is $d/\sqrt{(a^2 + b^2 + c^2)}$.

Directed straight lines drawn from the origin and passing through the points (1, 2, 2), (2, 2, 1) and (2, 1, 2) meet the sphere $x^2 + y^2 + z^2 = a^2$ in the points P, Q, R. Find the equation of the plane PQR and show that its perpendicular distance from the origin is $5a/3\sqrt{3}$.

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CHAPTER XIII

The General Conicoid—Sphere, Ellipsoid, etc.

The General Conicoid

1. The general equation of the conicoid is

$$f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0.$$

The tangent plane at (x_1, y_1, z_1) is

$$(x - x_1)\left(\frac{\partial f}{\partial x}\right)_1 + (y - y_1)\left(\frac{\partial f}{\partial y}\right)_1 + (z - z_1)\left(\frac{\partial f}{\partial z}\right)_1 = 0,$$

where $\left(\frac{\partial f}{\partial x}\right)_1$ is the result of replacing x, y, z by x_1, y_1, z_1 respectively in $\frac{\partial f}{\partial x}$, etc.

The rule for writing down this tangent plane is: Replace x^2 by xx_1, y^2 by yy_1, z^2 by zz_1, $2yz$ by $(yz_1 + zy_1)$, $2zx$ by $(zx_1 + xz_1)$, $2xy$ by $(xy_1 + yx_1)$, $2x$ by $(x + x_1)$, $2y$ by $(y + y_1)$, $2z$ by $(z + z_1)$, and leave the constant term unchanged.

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The polar plane of (α, β, γ) with respect to the conicoid $f(x, y, z) = 0$ is

$$(x - \alpha)\left(\frac{\partial f}{\partial x}\right)_{\alpha, \beta, \gamma} + (y - \beta)\left(\frac{\partial f}{\partial y}\right)_{\alpha, \beta, \gamma} + (z - \gamma)\left(\frac{\partial f}{\partial z}\right)_{\alpha, \beta, \gamma} = 0,$$

i.e. the same equation as for the tangent plane at (x_1, y_1, z_1) with x_1, y_1, z_1 replaced by α, β, γ respectively. [(α, β, γ) is not generally on the surface].

The equation of the plane containing all chords of the conicoid $f(x, y, z) = 0$ bisected at (x_1, y_1, z_1) is

$$(x - x_1)\left(\frac{\partial f}{\partial x}\right)_1 + (y - y_1)\left(\frac{\partial f}{\partial y}\right)_1 + (z - z_1)\left(\frac{\partial f}{\partial z}\right)_1 = 0,$$

but (x_1, y_1, z_1) does *not* lie on the conicoid.

The locus (diametral plane) of mid-points of chords of the conicoid $f(x, y, z) = 0$ parallel to $x/l = y/m = z/n$, is given by

$$l\frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y} + n\frac{\partial f}{\partial z} = 0,$$

and a diametral plane is a principal plane if it is perpendicular to the chords it bisects.

The Sphere

2. The equation of a sphere centre (x_1, y_1, z_1) radius r is

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2,$$

and if the centre is O the equation is

$$x^2 + y^2 + z^2 = r^2.$$

The general equation of a sphere is

$$S(x, y, z) \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$$

Its centre is $(-u, -v, -w)$ and radius $\sqrt{(u^2 + v^2 + w^2 - d)}$.

The tangent plane to this sphere at (x_1, y_1, z_1) is

$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0.$$

The polar plane of (α, β, γ) to this sphere is

$$x\alpha + y\beta + z\gamma + u(x + \alpha) + v(y + \beta) + w(z + \gamma) + d = 0.$$

If t be the length of tangent from (x_1, y_1, z_1) to the sphere $S(x, y, z) = 0$, then

$$t^2 = S(x_1, y_1, z_1).$$

The spheres

$$S_1 \equiv x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0,$$
$$S_2 \equiv x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$$

will cut orthogonally if

$$2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2.$$

Any sphere through the circle of intersection of $S_1 = 0$ and $S_2 = 0$ is given by

$$S_1 + \lambda S_2 = 0,$$

which becomes the radical plane when the constant $\lambda = -1$.

If $P = 0$ represents a plane, then the equation $S + \lambda P = 0$ represents a sphere through the circle of intersection of the sphere $S = 0$ and the plane $P = 0$.

The sphere on the line joining (x_1, y_1, z_1), (x_2, y_2, z_2) as diameter has as its equation

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0.$$

The Ellipsoid

3. The standard equation of the ellipsoid whose centre is the origin and axes the axes of co-ordinates is

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$$

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The equation of the tangent plane at (x_1, y_1, z_1) to this ellipsoid is

$$xx_1/a^2 + yy_1/b^2 + zz_1/c^2 = 1,$$

and the normal at this point is

$$\frac{x - x_1}{x_1/a^2} = \frac{y - y_1}{y_1/b^2} = \frac{z - z_1}{z_1/c^2}.$$

Using the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$; the polar plane of (α, β, γ) is $\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} = 1$.

The equation of the plane containing all chords bisected at (α, β, γ) is

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}.$$

The equation of the diametral plane bisecting all chords having direction cosines l, m, n is

$$\frac{lx}{a^2} + \frac{my}{b^2} + \frac{nz}{c^2} = 0.$$

The condition that the plane $lx + my + nz = p$ shall touch the ellipsoid is

$$a^2l^2 + b^2m^2 + c^2n^2 = p^2.$$

The Hyperboloids

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4. The equation of the hyperboloid of one sheet is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

The equation of the hyperboloid of two sheets is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

The Paraboloid

5. The standard equation of the paraboloid is

$$\frac{x^2}{a} + \frac{y^2}{b} = 2z,$$

and this paraboloid is elliptic if a and b have the same sign, and hyperbolic if a and b have opposite signs.

6. The tangent plane at (x_1, y_1, z_1) to *any* surface $F(x, y, z) = 0$ is given by

$$(x - x_1)\left(\frac{\partial F}{\partial x}\right)_1 + (y - y_1)\left(\frac{\partial F}{\partial y}\right)_1 + (z - z_1)\left(\frac{\partial F}{\partial z}\right)_1 = 0.$$

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EXAMPLES XIII

1. Show that $x^2 + y^2 + z^2 - 2ux - 2vy - 2wz + c = 0$ is the equation of a sphere. Find its radius and the co-ordinates of its centre.

   Find the polar plane of P (x_1, y_1, z_1) with respect to this sphere and show that, if P describes a straight line (L), then the polar plane passes through a fixed line (L_1) perpendicular to (L).

2. Two spheres of equal radius a, lie in the positive octant of the co-ordinate frame. One sphere touches the three co-ordinate axes and the other touches the three co-ordinate planes.

   Obtain the equations of the two spheres.

   Prove that the circle in which the two spheres intersect lies in the plane $x + y + z = 3a(2 + \sqrt{2})/4$, and find the co-ordinates of the centre and the radius of this circle.

3. Find the centre and radius of the circle of intersection of the sphere

$$x^2 + y^2 + z^2 + 12x - 12y - 16z + 111 = 0,$$

   and the plane $2x + 2y + z - 17 = 0$.

   Show that there exist two planes through the origin which meet the above plane at right angles and touch the sphere.

4. Find the equation of the sphere which has its centre at the point $(2, 3, -1)$ and touches the line

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$$\frac{x - 13}{10} = \frac{y - 8}{3} = \frac{z + 7}{-8}.$$

   Find also the equation of the tangent plane to the sphere which contains the above tangent line.

5. Three spheres have centres $(0, 0, 0)$, $(3a, 0, 0)$, $(0, 4a, 0)$ and radii a, $2a$, $3a$ respectively, and two planes making an acute angle ϕ with each other, are such that every one of the spheres touches the two planes. Show that $\cos\phi = 5/18$.

6. A straight line L passes through the origin and is a tangent to each of the spheres

$$x^2 + y^2 + z^2 + 2ax + p = 0, \quad x^2 + y^2 + z^2 + 2by + q = 0.$$

   Show that the angle ϕ between L and the z-axis is given by

$$\sin^2\phi = (p/a^2) + (q/b^2).$$

   Also find the distance between the points of contact of L with the spheres.

7. Find the equation of the sphere with centre $(a, 2a, 4a)$ which touches the line $2x = y = z$.

   Find also the radius and centre of the circle in which the polar plane of the origin with respect to the above sphere cuts the sphere.

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8. Prove that the distance of the point (α, β, γ) from the plane $lx + my + nz = k$ is r where

$$r^2(l^2 + m^2 + n^2) = (l\alpha + m\beta + n\gamma - k)^2.$$

Find the equations of the planes which touch the three spheres

$$x^2 + y^2 + z^2 - 2x = 0, x^2 + y^2 + z^2 - 2y = 0,$$
$$x^2 + y^2 + z^2 - 2z = 0. \quad (II)$$

9. Write down a condition that the sphere with centre (a, b, c) and radius R should be orthogonal to the sphere with centre (a_1, b_1, c_1) and radius R_1.

Spheres have their centres on the line

$$(x - a)/l = (y - b)/m = (z - c)/n$$

and are orthogonal to the sphere with centre (a, b, c) and radius R. Show that these spheres meet the plane $z = 0$ in a system of co-axial circles whose radical axis is the line

$$l(x - a) + m(y - b) - nc = 0, \quad z = 0.$$

10. Find the condition that the spheres

$$x^2 + y^2 + z^2 + 2ax + 2by + 2cz + d = 0,$$
$$x^2 + y^2 + z^2 + 2a_1x + 2b_1y + 2c_1z + d_1 = 0$$

should be orthogonal.

Prove that there exist eight spheres of radius five units which are orthogonal to the sphere $x^2 + y^2 + z^2 = 16$, touch the x-axis and cut off a segment of length two units from the y-axis.

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11. Obtain the equations, in their simplest form, of two non-intersecting straight lines, the angle and the shortest distance between them being 2α and $2c$ respectively.

Prove that the locus of the centres of spheres which touch two given non-intersecting straight lines is a conicoid cutting, in general, any plane perpendicular to their shortest distance in a rectangular hyperbola.

12. Find the equations of the tangent plane and normal to the ellipsoid $ax^2 + by^2 + cz^2 = 1$ at the point (x_1, y_1, z_1).

If the tangent plane to this ellipsoid at the point P intersects the plane $z = 0$ in a line whose distance from the origin is constant, show that the locus of the point of intersection of the normal at P with the plane $z = 0$ is an ellipse.

13. Prove that the plane $lx + my + nz = 1$ touches the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at the point P (a^2l, b^2m, c^2n) where $a^2l^2 + b^2m^2 + c^2n^2 = 1$.

The normal to the ellipsoid at P meets the plane through the origin parallel to the tangent plane at P at Q. Show that the co-ordinates of Q are

$$\{(a^2 - \theta)l, (b^2 - \theta)m, (c^2 - \theta)n\}, \text{ where } \theta(l^2 + m^2 + n^2) = 1.$$

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If the tangent plane to the ellipsoid at P also touches the sphere $x^2 + y^2 + z^2 = r^2$, show that Q lies on the ellipsoid

$$\frac{x^2}{(a^2 - r^2)^2} + \frac{y^2}{(b^2 - r^2)^2} + \frac{z^2}{(c^2 - r^2)^2} = \frac{1}{r^2}.$$

14. P is the point (X, Y, Z) and P_1 is the point (aX, bY, cZ). Prove that, when P describes the sphere $x^2 + y^2 + z^2 = 1$, P_1 describes an ellipsoid.

O is the origin and OA, OB, OC are three mutually perpendicular radii of the sphere. A_1, B_1, C_1 are the positions of P_1 which correspond to the positions A, B, C of P respectively. Prove that OA_1, OB_1, OC_1 are conjugate semi-diameters of the ellipsoid and that the centroid of A_1, B_1, C_1 lies on another fixed ellipsoid whose principal axes coincide in position with those of the first ellipsoid.

15. Prove that the middle points of a set of parallel chords of an ellipsoid are coplanar.

If the equation of the ellipsoid is $7x^2 + 3y^2 + 4z^2 = 1$ and the direction cosines of the chords are given by $l : m : n = 2 : 5 : 4$, find the inclination of the chords to the plane that bisects them.

16. Prove that all chords of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, which are bisected at the point (α, β, γ), lie in the plane

$$\frac{\alpha(x - \alpha)}{a^2} + \frac{\beta(y - \beta)}{b^2} + \frac{\gamma(z - \gamma)}{c^2} = 0.$$

Prove also that the centre of the conicoid which the above ellipsoid is cut by any plane through the point (a, b, c) lies on the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x}{a} + \frac{y}{b} + \frac{z}{c}.$$

17. Prove that there are only two points on the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$$

at which the normals are equally inclined to the positive directions of the co-ordinate axes and find their co-ordinates.

The normal, equally inclined to the positive directions of the co-ordinate axes, is drawn from a point in the positive octant on the ellipsoid $3x^2 + 4y^2 + 12z^2 = 24$.

Prove that the polar planes of all points on this normal pass through the line

$$\frac{x - 16}{8} = \frac{y + 12}{-9} = z.$$

18. Find the equations of the normal at the point (α, β, γ) on the conicoid $ax^2 + by^2 + cz^2 = 1$.

Prove that, if the normal at a point on this conicoid touches the sphere $x^2 + y^2 + z^2 = r^2$ the point must also lie on the surface

$$(x^2 + y^2 + z^2 - r^2)(a^2x^2 + b^2y^2 + c^2z^2) = 1.$$

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19. Prove that the plane which cuts the conicoid $ax^2 + by^2 + cz^2 = 1$ in a conic with centre (α, β, γ) is

$$a\alpha(x - \alpha) + b\beta(y - \beta) + c\gamma(z - \gamma) = 0.$$

Planes passing through the line $x - 2 = y = z$ cut the conicoid $4x^2 + 3y^2 + z^2 = 1$; prove that the locus of the centres of the plane sections so formed is a conic.

By transferring the origin of co-ordinates to the point (1, 0, 0), or otherwise, find the position of the centre of this conic.

20. Show that, for all values of the constants λ and μ the lines

$$\frac{x + a}{z + y} = \frac{z - y}{x - a} = \lambda, \quad \frac{x + a}{z - y} = \frac{z + y}{x - a} = \mu,$$

lie on the surface $x^2 + y^2 - z^2 = a^2$.

Prove that, if these lines intersect at right angles, $\lambda = -\mu$, and the locus of their point of intersection, for different values of λ, is a circle in the plane $z = 0$.

21. Define the polar plane of a point with respect to the surface

$$ax^2 + by^2 + cz^2 = 1.$$

Show that the polar plane of a variable point on the line $(x - f)/l = (y - g)/m = (z - h)/n$ with respect to the surface passes through a fixed straight line, and find the equations of this line.

22. Two planes which intersect in a line L touch a conicoid at the points P and Q. Prove that the polar plane of any point on PQ passes through the line L.

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Show that the distance between the points of contact of the two tangent planes to the conicoid $x^2 + y^2 - 2z^2 = 1$ passing through the line $\frac{x + 1}{4} = \frac{y + 1}{12} = \frac{z + 1}{9}$ is $5\sqrt{2}$ units.

23. Prove that the plane $lx + my + nz = p$ cuts the conicoid

$$ax^2 + by^2 + cz^2 = 1$$

in a conic whose centre (α, β, γ) is given by the equations

$$a\alpha/l = b\beta/m = c\gamma/n = (a\alpha^2 + b\beta^2 + c\gamma^2)/p.$$

Deduce that the plane touches the ellipsoid if

$$l^2/a + m^2/b + n^2/c = p^2.$$

Find the locus of the centres of sections of the conicoid $ax^2 + by^2 + cz^2 = 1$ made by the tangent planes to the conicoid $Ax^2 + By^2 + Cz^2 = 1$.

24. Show that the locus of all points on the conicoid

$$ax^2 + by^2 + cz^2 = 1$$

through which a tangent line to the conicoid can be drawn with given direction ratios $l : m : n$ is the conic obtained as the intersection of the conicoid and the plane $alx + bmy + cnz = 0$.

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Tangents are drawn to the sphere $x^2 + y^2 + z^2 = 1$ which have direction ratios $1 : 1 : 2$. Show that the locus of the mid-points of the segments intercepted on these tangents by the ellipsoid $2x^2 + y^2 + z^2 = 2$ is an ellipse whose semi-axes are of lengths $1, \frac{3}{7}\sqrt{6}$ respectively.

25. Find the equations to the tangent plane and normal to the surface $xyz = a^3$ at the point $P(x_1, y_1, z_1)$.

If the tangent plane cuts the axes at L, M, N show that P is at the intersection of the medians of the triangle LMN and that the volume of the tetrahedron OLMN (where O is the origin) is constant.

If the normal cuts the co-ordinate planes $x = 0$, $y = 0$, $z = 0$ at A, B, C respectively, show that $PA : PB : PC = x_1^2 : y_1^2 : z_1^2$.

26. Prove that the point $P(\theta^3 + 3\lambda\theta^2, \theta^2 + 2\lambda\theta, \theta + \lambda)$ lies on the surface $(yz - x)^2 = 4(y - z^2)(xz - y^2)$, and show that the equation of the tangent plane at P is $x - 3y\theta + 3z\theta^2 - \theta^3 = 0$.

Deduce that, if θ remains constant but λ varies, the locus of P is a line at every point of which the surface has the same tangent plane.

27. Prove that the plane $y = \lambda x$ meets the surface $z = f(y/x)$ in a straight line L_λ perpendicular to the z-axis.

$P(x_1, y_1, z_1)$ is a point on the surface $z = \frac{y}{x}\left(\frac{y^2}{x^2} - 1\right)$, and it does not lie in the plane $x = 0$.

Find the equation of the normal at P and prove that, if the normal at P is perpendicular to the line L_λ, then P lies either on L_λ or on one of two fixed lines independent of λ.

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28. Prove that all points on the line $x = y$, $z = 0$ lie on the surface $(y - 1)(z - c)^2 - (x - 1)(z + c)^2 = 0$, and that the normals to the surface at these points lie on the surface

$$(x - y)(x + y - 2) - cz = 0.$$

29. A variable line PQ intersects the axis OZ in Q, and the angle PQZ has the constant value α. The variation is such that the length OQ, λ, is a given function of the angle, β, between the planes PQZ, XOZ. Prove that PQ sweeps out a surface whose equation has the form $z = r \cot \alpha + f(y/x)$ where $r = \sqrt{(x^2 + y^2)}$ and f is a function whose form is determined solely by the relation connecting β and λ.

Show that any such surface satisfies the equation

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = r \cot \alpha.$$

30. Find the equation of the tangent plane at the point

$$\{-2a/(1 + \lambda), -2a/(1 - \lambda), a/2\}$$

of the surface whose equation is $1/x + 1/y + 1/z = 1/a$.

Show that the intersection of the tangent plane with the plane $z = 0$ touches the hyperbola given by $xy = a(x + y)$, $z = 0$.

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31. Prove that the equation of the tangent plane to the surface $F(x, y, z) = 0$ at the point (α, β, γ) is

$$(x - \alpha)\frac{\partial F}{\partial \alpha} + (y - \beta)\frac{\partial F}{\partial \beta} + (z - \gamma)\frac{\partial F}{\partial \gamma} = 0.$$

Find the equation of the tangent plane at any point (α, β, γ) on the surface $x^2y - a^2z = 0$, and prove that the tangent planes at all points of the surface which lie in the plane $x = \alpha$ intersect the plane $x = 0$ in parallel lines.

32. Prove that there is one point P on the surface

$$xyz - a^2x - b^2y - c^2z + 2abc = 0$$

at which there is no unique tangent plane.

The surface cuts the x- and y-axes at Q and R respectively. Prove that the plane PQR is a tangent plane at the point $\left(\frac{bc}{a}, \frac{ca}{b}, 0\right)$ and cuts the surface in the line of intersection of the planes $z = 0$, $a^2x + b^2y = 2abc$.

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CHAPTER XIV

Area under a Curve, Volume of Revolution, etc.

Areas, etc.

1. The area under the curve $y = f(x)$ between $x = a$ and $x = b$ is given by

$$\underset{\delta x \to 0}{\mathrm{Lt}} \sum_{x=a}^{x=b} y\delta x = \int_a^b y dx.$$

The mean value of y between $x = a$ and $x = b$ is

$$\frac{1}{b-a}\int_a^b y dx.$$

The area between the curve $r = f(\theta)$ and the radii vectors given by $\theta = \theta_1$ and $\theta = \theta_2$ is given by

$$\int_{\theta_1}^{\theta_2} \tfrac{1}{2} r^2 d\theta = \tfrac{1}{2}\int_{\theta=\theta_1}^{\theta=\theta_2} (xdy - ydx).$$

If $f(x)$ be an even function of x

$$\int_{-a}^{+a} f(x)dx = 2\int_0^{+a} f(x)dx,$$

and if $f(x)$ be an odd function of x

$$\int_{-a}^{+a} f(x)dx = 0.$$

Volumes and Surfaces

2. If the area under $y = f(x)$ between $x = a$ and $x = b$ be rotated through one complete revolution about OX the volume generated is

$$\int_a^b \pi y^2 dx,$$

and the surface generated is

$$\int_{x=a}^{x=b} 2\pi y ds.$$

If the area between the curve and the y-axis from $y = c$ to $y = d$ be rotated through one complete revolution about OY the volume generated is

$$\int_c^d \pi x^2 dy,$$

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and the surface generated is

$$\int_{y=c}^{y=d} 2\pi x ds.$$

Lengths of Curves

3. The length of curve between the points (a, c) and (b, d) on the curve $y = f(x)$ is

$$\int_{x=a}^{x=b} ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

For the curve given in the parametric form the length of curve is

$$\int \sqrt{(\dot{x}^2 + \dot{y}^2)}dt.$$

For the polar equation $r = f(\theta)$ the length of curve is

$$\int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Centres of Gravity

4. If $(\bar{x}, \bar{y}, \bar{z})$ be the centre of gravity of a solid body

$$\bar{x} = \frac{\int x dw}{\int dw}, \quad \bar{y} = \frac{\int y dw}{\int dw}, \quad \bar{z} = \frac{\int z dw}{\int dw}.$$

For centroid replace weight w by volume V for solid body, and replace w by A (area) for laminas, etc.

Pappus' (or *Guldinus'*) *First Theorem* is :—

If a closed area be rotated through any angle, not greater than 2π radians, about a line in its plane that does not intersect it, the volume generated is equal to the product of the area and the length of path of the centroid.

Pappus' (or *Guldinus'*) *Second Theorem* is :—

If a plane curve be rotated through an angle, not greater than 2π radians, about a line in its plane not intersecting it, the surface generated is equal to the product of the length of the curve and the length of the path of the centroid.

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EXAMPLES XIV

1. Sketch the curves $y = \frac{6}{5}\sqrt{(25 - x^2)}$ and $y = 3 + \frac{3}{5}\sqrt{(25 - x^2)}$ where the square roots are taken positively.

   Show that the area in the first quadrant bounded by the curves and the line $x = 5$ is $15(1 - \pi/4)$, and find the volume generated when this area is rotated through four right angles about the axis OY.

2. Prove that the area bounded by the line $y = 1$ and the curve $y = \tanh x$ in the region defined by $x \geqslant 0$ is $\log_e 2$.

3. Sketch the graph of the curve given by the equation

$$y^2(1 - x)(x - 2) = x^2,$$

and prove that the area between the curve and its asymptotes is 3π.

4. Prove that, if the rectangular Cartesian co-ordinates of a point on a closed curve are given as functions of a parameter t, the area enclosed is $\frac{1}{2}\int\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$ taken between the appropriate limits.

Prove that the curve $x = \cos^3 t$, $y = 2\sin^3 t$ is of length $28/3$ and encloses an area $3\pi/4$.

5. If s is the length of arc of a plane curve whose Cartesian equation is $y = f(x)$, prove that

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{df}{dx}\right)^2.$$

Prove that the length of the arc of the curve

$$y = (x + 1)(x + 2) - \tfrac{1}{8}\log_e(2x + 3)$$

between the points for which $x = 1$ and $x = 2$, respectively is

$$6 + \tfrac{1}{8}\log_e(7/5).$$

6. The co-ordinates of a variable point are given by

$$x = 3a\cos\theta - a\cos 3\theta,\ y = 3a\sin\theta - a\sin 3\theta.$$

Show that, when θ increases from 0 to α (< π), the point describes a curve of length $6a(1 - \cos\alpha)$, and that the radius of curvature of the curve at the point $\theta = \alpha$ is $3a\sin\alpha$.

7. Draw a sketch of the curve $y^2 = x^3 - 2x$, showing that it consists of a closed oval S and an open branch S_1.

If the area bounded by S and the area bounded by S_1, and the line $x = +(2 + \frac{4}{3}\sqrt{3})^{\frac{1}{2}}$ are rotated about the axis of x through two right angles, prove that the separate volumes swept out are in the ratio 3 : 4.

8. A is the vertex and LL′ ($= 4a$) the latus rectum of a parabola. Find the area generated when the arc AL is rotated through four right angles (i) about the axis of the parabola, (ii) about LL′.

9. A cylindrical hole is drilled through a hemispherical casting of radius R, the axis of the hole being along the axis of the hemisphere; find the length of the cylindrical wall of the hole when its volume is a maximum, and determine the metal remaining in this case. (*I*)

10. O is the pole and OX the initial line of polar co-ordinates. The straight line $r\cos\theta = 2$ intersects OX at A and the curve $r = 3 + 2\cos\theta$ at B and C. Prove that the area of the triangle OAB is $2\sqrt{3}$.

Find also the area of the smaller of the two portions into which the line BC divides the area enclosed by the curve $r = 3 + 2\cos\theta$.

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11. (i) Show that the area bounded by the curve whose equation in polar co-ordinates is $r = f(\theta)$, and the two radii vectors $\theta = \theta_1$ and $\theta = \theta_2$, where $0 < \theta_2 - \theta_1 < 2\pi$ is $\frac{1}{2}\int_{\theta_1}^{\theta_2} \{f(\theta)\}^2 \, d\theta$.

(ii) Prove that the parabola $y^2 = x$ divides the circle $x^2 + y^2 = 2$ into two portions whose areas are in the ratio $(9\pi - 2) : (3\pi + 2)$.

12. The equation of a plane curve is $x = f(\theta)$, where x is a Cartesian co-ordinate and (r, θ) are polar co-ordinates and $x = r\cos\theta$ and $y = r\sin\theta$.

Prove that the length of the curve included between the radii vectors $\theta = \theta_1$ and $\theta = \theta_2$ is

$$\int_{\theta_1}^{\theta_2} [\{f'(\theta) + f(\theta)\tan\theta\} + \{f(\theta)\}^2]^{\frac{1}{2}} \sec\theta d\theta.$$

Verify the correctness of the formula by finding, in the following simple cases,

(i) the length of the curve $x = a\cot\theta$ between $\theta = \theta_1$, and $\theta = \theta_2$, and

(ii) the length of the closed curve $x = a\cos\theta$.

13. Starting from the point whose polar co-ordinates are (a, o) a point P moves in the plane in such a way that the direction of its motion always makes the same angle $\pi/4$ with the radius vector OP.

Prove that the equation of the locus of P is $r = ae^{\theta}$.

Prove that the difference of the areas described by the radius vector OP as θ increases (i) from $2(n - 1)\pi$ to $2n\pi$, (ii) from $2n\pi$ to $2(n + 1)\pi$, is $a^2 e^{4n\pi} \sinh^2 2\pi$.

Shade on a sketch the area so calculated when $n = 2$.

14. If $r = f(\theta)$ and $r = g(\theta)$ are two closed oval curves surrounding the origin, and the second lies wholly inside the first, show that the area of the annular region enclosed between them is

$$\frac{1}{2}\int_0^{2\pi} [\{f(\theta)\}^2 - \{g(\theta)\}^2] d\theta.$$

If P is a variable point of the ellipse $l/r = 1 - e\cos\theta$ and Q is a point on the radius vector OP, on the side of P remote from O, such that PQ is of constant length c, prove that the area between the ellipse and the locus of Q is $\pi c(2b + c)$, where $2b$ is the length of the minor axis of the ellipse.

15. Sketch roughly the curve whose polar equation is $r = a\tanh\frac{1}{2}\theta$.

Find the length s of the arc and the area A of the sector measured from the radius vector at $\theta = 0$ to the radius vector (r, θ) and show that $2A = a(s - r)$.

16. The area enclosed by the lemniscate $r^2 = a^2\cos 2\theta$ is rotated through two right angles about the line $\theta = \frac{1}{2}\pi$. Prove that the volume of the solid so formed is $\frac{\sqrt{2}}{8}\pi^2 a^3$.

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17. Draw the curve whose equations are $x = a(t-1)^2$, $y = at^2$, for values of t in the range $-2 \leqslant t \leqslant 2$.

Find the y co-ordinate of the centroid of the area enclosed by the curve and the line $y = 4a$. (*I*)

18. Show that the area between the parabola $y^2 = 4x$ and the straight line $y = 2x - 4$ is 9.

Find the co-ordinates of the centroid of this area. (*I*)

19. Find the area included between the parabola $y^2 = x$ and the straight line $x - 5y + 6 = 0$. Determine the co-ordinates of the centroid of this area. (*I*)

20. Find the area included between the straight line $y = x - 1$ and the curve $y = x^3 - 5x^2 + 8x - 4$.

Find also the x-co-ordinate of the centroid of this area. (*I*)

21. Find the position of the centre of gravity of a piece of uniform wire in the form of an arc of a circle.

Use this result, or otherwise, to find the position of the centroid of a semi-circular area.

A semi-circle ABC, radius a, revolves about a line parallel to, and at a distance b from the bounding diameter AC, so that the point B of the circumference of the area is at a distance $(a + b)$ from the axis of rotation. Find (i) the area of the surface, and (ii) the volume of the solid formed. (*I*)

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22. ABCD is a square of side 8 inches, E is the mid-point of CD and F is the mid-point of DA. A circular quadrant centre C and radius CE is drawn on the square. Find the distance from AB of the centre of gravity of a thin uniform plate of metal in the shape left when the circular quadrant and the triangle DEF are removed from the square. (*I*)

23. (*a*) Evaluate (i) $\displaystyle\int_{\pi/6}^{\pi/4} \frac{dx}{\sin^4 x \cos^4 x}$, (ii) $\displaystyle\int_{1/2}^{2/3} \frac{dx}{x\sqrt{(3x - 2x^2 - 1)}}$.

(*b*) Find the centroid of a sector of a circle when the surface density varies as the distance from the centre. (*I*)

24. (i) State and prove Pappus's theorem on the volume of a solid of revolution.

(ii) The area bounded by $y = 0$, $x = 0$, $y = \cos x$ $(0 \leqslant x \leqslant \pi/2)$ is rotated about the line $x = \frac{1}{2}\pi$. Prove that the volume swept out in one revolution is 2π. (*I*)

25. Prove the theorem of Pappus relating to the surface obtained by rotating a curve about an axis in its own plane.

A circle of radius a is rotated about an axis in its own plane distant b $(> a)$ from its centre so as to form a closed ring.

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The surface of the ring is divided into two parts by a circular cylinder of radius $b - a \cos \theta$ co-axial with the ring. Show that the areas of the two parts are

$$4a\pi[b\theta - a \sin \theta] \text{ and } 4a\pi[b(\pi - \theta) + a \sin \theta]. \qquad (I)$$

26. A semi-circular area radius a is rotated about an axis in its plane distant d from the centre of the straight edge and inclined at 60° to it.

If the axis of rotation does not cut the area and the latter is on the side of the straight edge remote from the axis of rotation, find (*a*) the volume, and (*b*) the *total* surface area of the ring formed.

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CHAPTER XV

First-Order Differential Equations

Differential equations of the first order and first degree are of the general form (in x and y)

$$P\frac{dy}{dx} + Q = 0,$$

where P and Q are functions of x and y.

The variables x, y are separable if $P/Q = f(y)/\phi(x)$, and in this case the solution is

$$\int f(y)dy + \int \phi(x)dx = C.$$

The equation $P\frac{dy}{dx} + Q = 0$ is homogeneous if P and Q are homogeneous in x and y and of the same degree. The method of solution is to use

$$v = y/x.$$

The equation $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$ is reducible to the homogeneous type if $x = X + h$ and $y = Y + k$ be used, where (h, k) is the point of intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, but in the special case when $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ use $z = a_1x + b_1y$.

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The linear equation of the first order is of the general type

$$\frac{dy}{dx} + Py = Q,$$

where P and Q are functions of x only.

The integrating factor is $e^{\int P dx}$, and the solution is

$$y = e^{-\int P dx}\{(\int Qe^{\int P dx})dx + C\}.$$

The general equation of the form $\frac{dy}{dx} + Py = Qy^n$, where P and Q are functions of x only, can be reduced to the previous form by using

$$z = 1/y^{n-1}.$$

For solutions of $\frac{dy}{dx} = (ax + by + c)^n$ use the substitution

$$z = ax + by + c.$$

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If $f(x, y, c) = 0$, where c is an arbitrary constant represent a family of curves which has a differential equation of $\phi\left(x, y, \frac{dy}{dx}\right) = 0$, the orthogonal trajectories of this family are given by

$$\phi\left(x, y, -1\Big/\frac{dy}{dx}\right) = 0.$$

EXAMPLES XV

1. Solve the following equations :

   (i) $\frac{dy}{dx} = y \tan x + y^3 \tan x$, (ii) $(2x + y + 3)\frac{dy}{dx} = x + 2y + 3$.

2. Solve the equations :

   $$\text{(i) } \frac{dy}{dx} = 2x - y\,; \qquad \text{(ii) } \frac{dy}{dx} = \frac{2x - y}{x + 2y - 5}.$$

3. (i) Solve the equation $\frac{dy}{dx} + \frac{2x(x + y)}{x^2 + y^2} = 0$.

   (ii) Find the integral curves of the equation

   $$(dy/dx - y)e^x + 1 = 0.$$

   Show that, in general, every curve of the system has *either* a real point of inflexion, on the line $y = 0$, *or* a real point for which y is a minimum, on the curve $y = e^{-x}$.

4. (i) Solve the differential equation

   $$x^2 dy/dx + 1 + (1 - 2x)y = 0.$$

   (ii) Prove that the differential equation

   $$\mathrm{M}dx + \mathrm{N}dy = 0,$$

   where M and N are functions of x and y, is exact if

   $$\partial\mathrm{M}/\partial y = \partial\mathrm{N}/\partial x.$$

   Show that a constant α can be found so that $(x + y)^\alpha$ is an integrating factor of

   $$(4x^2 + 2xy + 6y)dx + (2x^2 + 9y + 3x)dy = 0,$$

   and hence integrate the equation.

5. Show how to solve the differential equation

   $$dy/dx + \mathrm{P}y = \mathrm{Q}y^n,$$

   where P and Q are functions of x only.

   Find the solution of the equation

   $$xy^2 dy/dx - 2y^3 = 2x^3,$$

   which is such that $y = 1$ when $x = 1$.

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6. (i) If a and b are arbitrary constants, find the second order differential equation whose solution is $y = a \sec x + b \tan x$.

(ii) If $dy/dx = (x + y)^2$ and $y = \frac{1}{2}$ when $x = \frac{1}{2}$, calculate, correct to three significant figures, the value of y when $x = 0{\cdot}7$.

7. Solve the differential equations :

$$\text{(i)}\ \frac{d^2y}{dx^2} + 4y = \cos 2x + x\,;$$

$$\text{(ii)}\ \frac{dy}{dx} = \frac{4x - 2y + 4}{2x + y - 2}\,;$$

$$\text{(iii)}\ (1 + 3x)\,dy/dx + (3 - 9x)y = 0.$$

8. Solve the differential equations :

(i) $dy/dx + y \cot x = y^2 \sin^2 x$; (ii) $dy/dx = 2xy/(x^2 + y^2)$;

(iii) $y\dfrac{dy}{dx} = \sin x e^{x+2y}$; (iv) $y + x(x + 1)\dfrac{dy}{dx} = x(x + 1)^2 e^{-x^2}$.

9. Solve the equations :

$$\text{(i)}\ x(1 - x^2)dy/dx + (1 - 2x - x^2)y = 1,$$

$$\text{(ii)}\ y^2 dy/dx = x^2 + xy - y^2.$$

10. Integrate the differential equations :

$$\text{(i)}\ dy/dx = (y - x)/(y + x)$$

$$\text{(ii)}\ dy/dx = y/x + y^2/x^2\,;$$

$$\text{(iii)}\ (3y^2 + 4x) + 2xy\,dy/dx = 0. \qquad (II)$$

11. Solve the differential equations :

$$\text{(i)}\ dy/dx - y \tan x = e^x\,;$$

$$\text{(ii)}\ \frac{dy}{dx} = \frac{x^2 + 2xy - y^2}{x^2 - 2xy - y^2}. \qquad (II)$$

12. Show that the curves whose equation is $(x + y - a)^2 = 4xy$, where a is a parameter, are a family of parabolas touching the co-ordinate axes.

Find the differential equation of the family, and find also the equation of their orthogonal trajectories.

13. P (x, y) is a point on a curve C. The tangent at P meets the axis OX at T and has gradient p. The normal at P meets the axis OY at G. Prove that the gradient of TG is $(x + py)/(y - px)$.

If, for all positions of P for which T, G are determinate, TG is perpendicular to OP, show that C is a circle of the system of circles touching OX at O.

Write down the differential equation of the system of orthogonal trajectories of this system of circles.

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14. Find the equation of the system of orthogonal trajectories of the family of curves whose equation in polar co-ordinates is

$$r = a(1 - \cos\theta),$$

where a is a parameter.

Sketch clearly a typical curve of each system.

15. P is any point on a plane curve, T is the foot of the perpendicular from P to the x-axis, and O is the origin of co-ordinates. The perpendiculars at P to the radius vector OP is met at Q by the line through O parallel to the tangent to the curve at P. Given that PQ = OT, find the equation of the curve if it passes through the point (1, 1).

16. The normal at any point P(x, y) on a certain curve meets the axes of x and y respectively at points Q and R on opposite sides of P and such that $RP/RQ = x^2/y^2$. Find the equation of the curve, given that it passes through the point (1, 1).

17. N is the foot of the ordinate at a point P on a plane curve, and T, U are the respective points at which the tangent and normal at P meet the axis of x. If TN − TU = 2ON, where O is the origin, find the differential equation of the curve, and hence show that the curve belongs to a family of parabolas.

Find the two members of the family which pass through the point (− 3/4, 1).

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18. OXY is a rectangular system of axes in a plane.

A constant length $2k$ is intercepted on OX between the tangent and normal at any point of a certain curve in the plane OXY.

Find the differential equation of this curve and integrate it. Show also that the curve must lie between the lines $y = \pm k$.

19. The tangent to a plane curve at a point P meets the x-axis at Q. Find the differential equation of the curve if OP = OQ, where O is the origin of co-ordinates, and hence find the family to which the curve belongs.

Determine also the orthogonal system.

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CHAPTER XVI

Second Order Differential Equations, etc.

Second Order Differential Equations

1. To solve the equation $d^2y/dx^2 = f(y)$, use $\dfrac{d^2y}{dx^2} = p\dfrac{dp}{dy}$, where

$$p = \frac{dy}{dx}\left(v \text{ for } \frac{dx}{dt}\right) \qquad = \frac{d}{dy}(\tfrac{1}{2}p^2).$$

If k_1 and k_2 are the roots of the auxiliary equation $k^2 + ak + b = 0$, the solution of the equation $d^2y/dx^2 + a\,dy/dx + by = 0$ is

(i) $y = Ae^{k_1x} + Be^{k_2x}$, for k_1, k_2 real and unequal.

(ii) $y = e^{rx}\{C\cos qx + D\sin qx\}$, if k_1, k_2 are complex and $k_1 = p + iq$, $k_2 = p - iq$.

(iii) $y = (A + Bx)e^{+x}$, if $k_1 = k_2 = r$.

If k_1 and k_2 are the roots of the auxiliary equation

$$k(k-1) + ak + b = 0,$$

the solution of the homogeneous differential equation

$$x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0 \text{ is}$$

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(i) $y = Ax^{k_1} + Bx^{k_2}$, for k_1, k_2 real and unequal.

(ii) $y = (A + B\log_e x)x^p$, if $k_1 = k_2 = p$.

2. Operators :

$Dy \equiv \dfrac{dy}{dx}$, D is an operator and $f(D)$ is a polynomial in D.

If $\phi(x)$ be a polynomial in x of degree n, $\dfrac{1}{f(D)}\cdot\phi(x)$ is obtained by expanding $\dfrac{1}{f(D)}$ in powers of D by the binomial theorem as far as D^n.

$$\frac{1}{f(D)}\cdot e^{ax+b} = \frac{1}{f(a)}e^{ax+b}.$$

The special case when $f(a) = 0$ is dealt with below.

$$\frac{1}{f(D)}Ve^{ax+b} = e^{ax+b}\cdot\frac{1}{f(D+a)}\cdot V,$$

where V is a function of x.

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When $f(a) = 0$ in previous case, use $V = 1$ in this result and obtain

$$\frac{1}{f(D)} \cdot 1 \cdot e^{ax+b} = e^{ax+b} \frac{1}{f(D+a)} \cdot 1.$$

$$\frac{1}{f(D^2)} \sin(ax+b) = \frac{1}{f(-a^2)} \sin(ax+b).$$

$$\frac{1}{f(D^2)} \cos(ax+b) = \frac{1}{f(-a^2)} \cos(ax+b).$$

$$\frac{1}{p+qD} \cdot \sin(ax+b) = \frac{p-qD}{p^2-q^2D^2} \cdot \sin(ax+b)$$

$$= \frac{p \sin(ax+b) - aq \cos(ax+b)}{p^2+a^2q^2},$$

$$\frac{1}{p+qD} \cdot \cos(ax+b) = \frac{p-qD}{p^2-q^2D^2} \cdot \cos(ax+b)$$

$$= \frac{p \cos(ax+b) + aq \sin(ax+b)}{p^2+a^2q^2},$$

where p, q are constants.

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$$\frac{1}{f(D^2)} \sinh(ax+b) = \frac{1}{f(a^2)} \sinh(ax+b).$$

$$\frac{1}{f(D^2)} \cosh(ax+b) = \frac{1}{f(a^2)} \cosh(ax+b), \text{ etc.}$$

3. The solution of $d^2y/dx^2 + a\, dy/dx + by = F(x)$ is given by (u is complementary function and v is the particular integral)

$$y = u + v,$$

where u is the general solution of

$$\frac{d^2u}{dx^2} + a\frac{du}{dx} + bu = 0,$$

and

$$v = \frac{1}{D^2 + aD + b} \cdot F(x).$$

4. To solve the general homogeneous equation

$$x^2 d^2y/dx^2 + ax\, dy/dx + by = F(x) \qquad (I)$$

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use the substitution $x = e^t$ giving

$$x\frac{dy}{dx} = \frac{dy}{dt}, \quad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}, \quad \text{and}$$

(1) reduces to

$$d^2y/dt^2 + (a - 1)dy/dt + by = \phi(t)$$

which has been dealt with in 3.

5. If $D \equiv d/dt$ the simultaneous equations

$$f_1(D)x + \phi_1(D)y = F_1(t),$$
$$f_2(D)x + \phi_2(D)y = F_2(t)$$

can be solved by eliminating x or y between the two equations as if they were algebraic, whichever is more convenient.

EXAMPLES XVI

1. (i) Solve the equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = e^x \sin 3x$.

   (ii) By means of the substitution $x = e^t$, or by any other means, solve the equation

$$x^2\frac{d^2y}{xd^2} - 5x\frac{dy}{dx} + 9y = x^3 + \log_e x.$$

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2. (i) Solve the equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = x^3e^{-x}$.

   (ii) Show that the equation $(px - y)(px - 2y) + x^3 = 0$, where p denotes dy/dx, may be reduced to Clairaut's form by means of the substitution $y = vx$ and hence find its general integral and singular solution.

3. (i) Solve the equation $a^2\frac{d^2y}{dx^2} - 2ab\frac{dy}{dx} + 4b^2y = 4b^3x^2$, where a and b are non-zero constants.

   (ii) Find the general integral and singular solution of the equation $y = px - \log p$, where p denotes dy/dx.

4. (i) Solve the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x}.$$

   (ii) If $x = e^\theta$ and D and Δ denote the operators $\frac{d}{dx}$ and $\frac{d}{d\theta}$ respectively, prove that $x^2D^2 = \Delta(\Delta - 1)$.

   Solve the differential equation

$$x^2\frac{d^2y}{dx^2} + y = x^2.$$

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5. Define the Complementary Function and Particular Integral for an equation of the form

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x).$$

Using the substitution $t = \sqrt{x}$, or otherwise, solve the equation

$$4x\frac{d^2y}{dx^2} + 2(1 - \sqrt{x})\frac{dy}{dx} - 6y = e^{-2\sqrt{x}}.$$

6. Show that the constant n may be chosen so that, by the substitution $y = x^{-2}z$, the differential equation

$$x^2\frac{d^2y}{dx^2} + 4x(x + 1)\frac{dy}{dx} + (8x + 2)y = \cos x$$

reduces to the form $\frac{d^2z}{dx^2} + a\frac{dz}{dx} + bz = \cos x$, where a and b are constant.

Hence, or otherwise, solve the given equation.

7. Solve the differential equations :

(i) $d^2y/dx^2 - 9y = \cosh 3x + x^2$;

(ii) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = xe^{3x}$.

8. Integrate the differential equations :

(i) $2\frac{d^2y}{dx^2} - 7\frac{dy}{dx} - 4y = 4$;

(ii) $\frac{d^2y}{dx^2} = 4x + e^x$. *(II)*

9. Solve the differential equations :

(i) $\frac{d^2y}{dx^2} + 9y = x^2 + x + 1$;

(ii) $2y\frac{d^2y}{dx^2} - 3\left(\frac{dy}{dx}\right)^2 - 4y^2 = 0$. *(II)*

10. Solve the differential equations :

(i) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{-x}\cos x$;

(ii) $\frac{dy}{dx} + y\sec^2 x = 3\sec^2 x \tan x$. *(II)*

11. By means of the substitution $y = zx^{-1/2}$ transform the differential equation

$$4x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + (4x^2 - 1)y = 0,$$

and obtain the solution for which $y = 0$ when $x = \frac{1}{2}\pi$, and $y = 1$ when $x = \pi$.

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12. (i) Solve the equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x \cos x$.

(ii) If $x = \cosh z$, prove that

$$(x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \frac{d^2y}{dz^2}.$$

Solve the equation $(x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x$.

13. Prove that, in general, at every point on any curve in the x, y plane

$$\frac{d^2x}{dy^2} + \left(\frac{dx}{dy}\right)^3 \frac{d^2y}{dx^2} = 0.$$

Transform the differential equation

$$\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 = (2x - y)\left(\frac{dy}{dx}\right)^3$$

so that y is the independent variable and x is the unknown function of y. Hence obtain the general solution of the given differential equation.

14. Prove that the partial differential equation

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial z^2}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$$

is satisfied by $z = x\phi(y/x) + y\psi(y/x)$, where ϕ and ψ denote arbitrary functions.

15. Solve the simultaneous differential equations :

$$d^2y/dx^2 + 15y + 3z + 30 = 0,$$
$$d^2z/dx^2 + 2y + 10z + 4 = 0. \qquad (II)$$

16. Solve the simultaneous differential equations :

$$\frac{d^2x}{dt^2} + x = \frac{dy}{dt}, \quad 4\frac{dx}{dt} + 2x = \frac{dy}{dt} + 2y.$$

If x, y are the Cartesian co-ordinates of a point in a plane, and if, when $t = 0$, $x = 0$, $y = 1$, $\frac{dx}{dt} = 2$, prove that the point lies on the parabola $(5x - 2y)^2 = 4(y - 2x)$. (*II*)

17. If $6\frac{d^2x}{dz^2} - 3\frac{dx}{dz} + 6x + \frac{dy}{dz} = z$,

and $\frac{d^2x}{dz^2} + 8x + y = 0$,

find x and y in terms of z.

If, when $z = 0$, $x = 11/36$, $y = -22/9$, $dx/dz = 1/6$, show that the locus of the point (x, y) is a straight line through the origin. (*II*)

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CHAPTER XVII

Double and Triple Integrals

1. To find $\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y)dx\,dy = \int_{y_1}^{y_2} dy \int_{x_1}^{x_2} f(x, y)dx$, first find $\int_{x_1}^{x_2} f(x, y)dx$ considering y *as being constant*, and then integrate the result with respect to y between the limits y_1 and y_2.

2. To find triple integral

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y, z)dx\,dy\,dz = \int_{z_1}^{z_2} dz \int_{y_1}^{y_2} dy \int_{x_1}^{x_2} f(x, y, z)dx$$

first find $\int_{x_1}^{x_2} f(x, y, z)dx$ considering both y and z as *constants;* next integrate the result with respect to y between y_1 and y_2 as limits considering z as a *constant;* finally integrate the latter result between z_1 and z_2 with respect to z.

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N.B.—x_1, x_2, y_1, y_2, z_1, z_2 need not necessarily be constants.

3. The area of a plane closed curve cut by a straight line in two points only is given by $\int_{y_1}^{y_2}\int_{x_1}^{x_2} dx\,dy$, where y_1, y_2 are the extreme values of y over the area and x_1, x_2 the two values for x at the extremities of an element of area parallel to OX.

The area is also given by $\int_{x'_1}^{x'_2}\int_{y'_1}^{y'_2} dy\,dx$, where x'_1, x'_2 are the extreme values of x over the area and y'_1, y'_2 the two values for y at the extremities of an element parallel to OY.

4. The volume of the cylinder between the plane $z = 0$ and the surface $z = f(x, y)$ is given by

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y)dx\,dy = \int_{x'_1}^{x'_2}\int_{y'_1}^{y'_2} f(x, y)dy\,dx,$$

where x_1, x_2, y_1, y_2; x'_1, x'_2, y_1', y'_2 have the same meaning as in (3) for the area in which the cylinder cuts the plane $z = 0$.

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The volume of a solid could also be given by

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} dx\, dy\, dz,$$

but this is seldom used.

5. If a plane curve be given in polar co-ordinates its area is

$$\int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2} r\, d\theta\, dr,$$

and the cylinder upon it whose upper surface has the equation

$$z = f(r,\ \theta)$$

has the volume

$$\int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2} zr\, d\theta\, dr.$$

6. In the case of the double integral $\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y)\, dx\, dy$ the area over which the integral is taken is given by the boundaries $x = x_1$, $x = x_2$; $y = y_1$, $y = y_2$, where x_1, x_2, y_1, y_2 are not necessarily constants.

In order to change the order of integration the result is

$$\int_{x'_1}^{x'_2}\int_{y'_1}^{y'_2} (x, y)\, dy\, dx,$$

where y'_1, y'_2 are the two values of y at the ends of an element of the above area parallel to OY and x'_1, x'_2 are the extreme values of x over this area.

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EXAMPLES XVII

1. Evaluate $\iint r^4 \cos^3 \theta\, dr\, d\theta$ taken over the interior of the circle $r = 2a \cos \theta$. *(II)*

2. Evaluate $\iint xy\, dx\, dy$ for the regions in the positive quadrant between the axes and (*a*) the circle $x^2 + y^2 = a^2$, (*b*) the straight line $x/a + y/b = 1$. *(II)*

3. Change the order of integration in the double integral

$$\int_0^a dy \int_0^y \frac{x\, dx}{\sqrt{\{(a^2 - x^2)(a - y)(y - x)\}}},$$

and hence evaluate the integral. *(II)*

4. Change the order of integration in

$$\int_0^a dx \int_0^x \frac{dy}{\{(a-x)(x-y)(a-y)(a+y)\}^{\frac{1}{2}}}.$$

Hence, or otherwise, evaluate the integral. *(II)*

5. Indicate by a diagram the region over which the double integral $\int_0^a dx \int_0^{bx/a} \sqrt{\{x^2+y^2\}} \, . \, dy$ is taken, and evaluate the integral.

Hence, or otherwise, evaluate $\int_0^a dx \int_0^a \sqrt{(x^2+y^2)} \, . \, dy$. *(II)*

6. Show, in a diagram, the region over which the repeated integral $\int_0^{\frac{1}{2}\pi} dy \int_0^y \cos 2y \sqrt{(1-k^2 \sin^2 x)} dx$ extends.

Change the order of integration and evaluate the integral. *(II)*

7. Indicate by a sketch the region over which the double integral $\int_0^a dx \int_x^a (x^2+y^2) \, dy$ is taken.

Write down the integral with the order of integration changed, and, hence or directly, evaluate it. *(II)*

8. Express as a double integral the volume of the cylinder $x^2+y^2=a^2$ for which x is positive, included between the planes $z=0$ and $z-x=0$, and hence determine the volume.

Indicate by a sketch the geometrical significance of the result of the first integration, and thus check this result. *(II)*

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9. Find the volume within the cylinder $x^2+y^2-4x=0$ cut off by the cylinder $z^2=4x$. *(II)*

10. The axes of two right circular cylinders of the same radius R intersect at right angles. Prove that the volume which is inside both cylinders is $16R^3/3$. *(II)*

11. Find the volume within the cylinder $x^2+y^2=a^2$, intercepted between the surface $x^2=4a(a-z)$ and the plane $z=0$. *(II)*

12. Show that the equation $(x^2+y^2+z^2+c^2-a^2)^2=4c^2(x^2+y^2)$ represents an anchor ring.

Sketch the surface showing how it lies with respect to the axes of reference.

Prove that the co-ordinates of any point on the surface may be expressed in the form $[(c+a\cos\theta)\cos\phi, (c+a\cos\theta)\sin\phi, a\sin\theta]$ and evaluate the surface area by calculating $\iint a(c+a\cos\theta)d\theta\, d\phi$. *(II)*

13. Sketch the curve $r=a(1+\cos\theta)$ and express as double integrals (i) the area enclosed by the curve, (ii) the first and second moments of this area about the line $\theta=\pi/2$. Find the centroid and radius of gyration about this line. *(II)*

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CHAPTER XVIII

Fourier Series

1. If $f(x)$ can be expanded in a Fourier series of period 2π, then the series for $f(x)$ is

$$\tfrac{1}{2}b_0 + \sum_{n=1}^{\infty} b_n \cos nx + \sum_{n=1}^{\infty} a_n \sin nx,$$

where

$$b_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx,$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx, \text{ in the interval } 0 < x < 2\pi,$$

or

$$b_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx, \; b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx,$$

$$a_n = \frac{1}{\pi}\int f(x)\sin nx\, dx, \text{ in the interval } -\pi < x < \pi.$$

N.B.—The end points of the interval have to be tested separately.

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2. If the periodic function $f(x)$ of period 2π has the following values,

$f_1(x)$ for $0 < x \leqslant c_1$, $f_2(x)$ for $c_1 \leqslant x \leqslant c_2$, $f_3(x)$ for $c_2 \leqslant x \leqslant c_3$, etc., then in the interval $0 < x < 2\pi$ the value of $f(x)$ is

$$\tfrac{1}{2}b_0 + \sum_{n=1}^{\infty} b_n \cos nx + \sum_{n=1}^{\infty} a_n \cos nx,$$

where

$$b_0 = \frac{1}{\pi}\left\{\int_0^{c_1} f_1(x)dx + \int_{c_1}^{c_2} f_2(x)dx + \int_{c_2}^{c_3} f_3(x)dx + \ldots\right\},$$

$$b_n = \frac{1}{\pi}\left\{\int_0^{c_1} f_1(x)\cos nx\, dx + \int_{c_1}^{c_2} f_2(x)\cos nx\, dx + \int_{c_2}^{c_3} f_3(x)\cos nx\, dx + \ldots\right\}.$$

$$a_n = \frac{1}{\pi}\left\{\int_0^{c_1} f_1(x)\sin nx\, dx + \int_{c_1}^{c_2} f_2(x)\sin nx\, dx + \int_{c_2}^{c_3} f_3(x)\sin nx\, dx + \ldots\right\}.$$

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3. If the periodic function $f(x)$ has a period of $2c$, its Fourier series is

$$\tfrac{1}{2}b_0 + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{c},$$

where $$b_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx,\ b_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx,$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx,$$

for the interval $-c < x < c$.

4. If the periodic function $f(x)$ of period $2c$ is (or is chosen as) an *odd* function of x in the interval $-c < x < c$ the Fourier series is, in the range $0 < x < c$, composed of sines only; i.e.

$$\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{c},$$

where $$a_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx.$$

If $f(x)$ is (or is chosen as) an *even* function of x of period $2c$ in the interval $-c < x < c$, the Fourier series for $f(x)$ in the range $0 < x < c$ is a series of cosines only; i.e. $\tfrac{1}{2}b_0 + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi x}{c}$, where

$$b_0 = \frac{2}{c}\int_0^c f(x)dx, \text{ and } b_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx.$$

EXAMPLES XVIII

1. If $f(x) = 0$ for $-\pi < x \leqslant 0$, and $f(x) = x$ for $0 \leqslant x < \pi$, prove that

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left\{\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right\}$$
$$+ \frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} \ldots \quad (II)$$

2. Show how to obtain the coefficients in the Fourier expansion of a function $f(x)$ which is given in the range $(-\pi, \pi)$.

Obtain the harmonic analysis of the function defined by $f(x) = c$ in the range $(0, \alpha)$, $f(x) = 0$ in the range (α, π), and $f(x) = f(-x)$. *(II)*

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3. If $f(x) = -f(-x)$ in the range $-\pi < x < \pi$, show that the Fourier series for $f(x)$ in this range contains no cosine terms.

If the above function is defined by $f(x) = x$ when $0 \leqslant x \leqslant \pi/2$, and $f(x) = \pi - x$ when $\pi/2 \leqslant x \leqslant \pi$, show that, in the range $-\pi \leqslant x \leqslant \pi$,

$$f(x) = \frac{4}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{\sin(2n+1)x}{(2n+1)^2}. \qquad (II)$$

4. A function of x, of period 2π, is zero for $0 \leqslant x \leqslant \pi/2$ and $\cos x$ for $\pi/2 \leqslant x \leqslant \pi$.

Expand the function in a series of cosines of multiples of x, valid when $0 \leqslant x \leqslant \pi$.

Sketch the graph of the sum of the series for values of x between 0 and 3π. *(II)*

5. A function $f(x)$ is equal to $\frac{1}{4}x^2$ in the interval $-\pi < x < \pi$, and is repeated with this period.

Prove that $f(x) = \pi^2/12 - \cos x + \dfrac{\cos 2x}{2^2} - \dfrac{\cos 3x}{3^2} + \ldots$

and find the term containing $\cos 2px$ when $x = \pi/2$. *(II)*

6. State the conditions as to symmetry and periodicity which must be satisfied by a function if it can be expanded in a Fourier series of (*a*) sines of multiples of x, (*b*) cosines of odd multiples of x.

The function $f(x)$ is defined in the range $0 < x < \frac{1}{2}\pi$ by $f(x) = \pi/2 - x$.

If it satisfies conditions (*b*) above, draw a graph of the function throughout a period and find its Fourier series. *(II)*

7. Prove that, in the range $0 \leqslant x < \pi$,

$$x = 2(\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \ldots). \qquad (II)$$

8. Show that the function of x which is equal to $x(\pi - x)$ in the range $0 \leqslant x \leqslant \pi$ can be replaced by a Fourier series containing sines of odd multiples of x only.

Sketch the graph of the function represented by the *series* for the range $-2\pi < x < 4\pi$.

Determine coefficients in the series including that of $\sin(2n-1)x$. *(II)*

9. Show that, if the Fourier series of period 2π for a function $f(x)$ contains sine terms only, the coefficient of $\sin nx$ is given by

$$\frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx.$$

If $f(x) = 0$ when $0 < x \leqslant \pi/2$ and unity when $\pi/2 \leqslant x < \pi$, find the coefficients of the Fourier sine series and draw the graph of $f(x)$ from $x = 0$ to $x = 2\pi$. *(II)*

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10. What conditions must be satisfied by a periodic function $f(x)$ of period 2π, if its Fourier series is to contain only sines of odd multiples of x?

Show that, if these conditions be satisfied, the coefficient of $\sin(2n+1)x$ is

$$b_{2n+1} = \frac{4}{\pi}\int_0^{\pi/2} f(x)\sin(2n+1)x\,dx.$$

If such a function is zero for $0 < x < \pi/3$, and unity for $\pi/3 < x < \pi/2$, determine the first four terms of its Fourier series. *(II)*

11. Show that, in the range $0 < x < \pi$, $\sin x$ can be replaced by the cosine Fourier series

$$\frac{4}{\pi}\left[\tfrac{1}{2} - \tfrac{1}{3}\cos 2x - \tfrac{1}{15}\cos 4x - \tfrac{1}{35}\cos 6x - \cdots - \frac{\cos 2px}{4p^2-1}\right].$$

Show by a sketch the function represented by the sum of the series for values of x from $-\pi$ to $+3\pi$. *(II)*

12. The function $f(x)$ is defined in the range $0 < x < \pi$ by the formulae:

$$f(x) = \tfrac{1}{3}\pi, \quad 0 < x < \tfrac{1}{3}\pi,$$
$$f(x) = 0, \quad \tfrac{1}{3}\pi < x < \tfrac{2}{3}\pi,$$
$$f(x) = -\tfrac{1}{3}\pi, \quad \tfrac{2}{3}\pi < x < \pi.$$

Obtain the expansion in a series of cosines of multiples of x. Show that the terms $\cos nx$ are absent when the remainder on dividing n by 6 is 0, 2, 3, or 4. *(II)*

13. If the function $f(x)$ can be expanded in the Fourier series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

show that

$$a_n + ib_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\,e^{inx}dx.$$

Hence, or otherwise, expand the function $x(x-1)(x-2)$ in a full range Fourier series, valid in the interval $(0, 2)$ giving the general term.

Illustrate your result by drawing a graph of the function represented by the series for values of x outside the range. *(II)*

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CHAPTER XIX

Spherical Trigonometry

The sphere here considered has radius r in each case.

1. (i) Areas of lunes subtending angles of A, B, C are $2r^2A$, $2r^2B$, $2r^2C$ respectively.

(ii) Area of spherical triangle ABC is $(A + B + C - \pi)r^2$.

(iii) Spherical excess for triangle ABC is $A + B + C - \pi = E$.

(iv) Area of spherical polygon of n sides is

$$\{\text{sum of angles} - (n-2)\pi\}\, r^2.$$

2. (i) Cosine rule

$$\cos a = \cos b \cos c + \sin a \sin b \cos A,$$
$$\cos b = \cos c \cos a + \sin c \sin a \cos B,$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

(ii) Sine rule

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

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3. If the spherical triangle $A_1B_1C_1$ is the polar triangle of spherical triangle ABC, then

$$a_1 = \pi - A, \quad b_1 = \pi - B, \quad c_1 = \pi - C;$$
$$A_1 = \pi - a, \quad B_1 = \pi - b, \quad C_1 = \pi - c.$$

Hence, from the cosine rule,

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}, \text{ etc.}$$

4. For the right angled spherical triangle ABC in which $A = 90°$:

(i) $\cos a = \cos b \cos c$.

(ii) $\cos B = \tan c/\tan a$, $\cos C = \tan b/\tan a$.

(iii) $\sin B = \sin b/\sin a$, $\sin C = \sin c/\sin a$.

(iv) $\tan B = \tan b/\sin c$, $\tan C = \tan c/\sin b$.

(v) $\cos B = \cos b \sin C$, $\cos C = \cos c \sin B$.

(vi) $\cos a = \cot B \cot C$.

(vii) Napier's Rule :—

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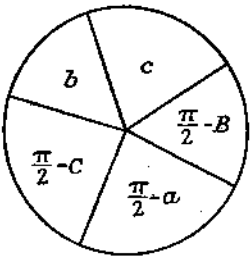
Draw the diagram as shown. Choose any quantity as the *middle part* which has then two adjacent quantities and two others named the opposite parts, then



(*a*) The sine of the middle part is equal to the product of the tangents of the adjacent parts.

(*b*) The sine of the middle part is equal to the product of the cosines of the opposite parts.

5. Napier's Analogies.

$$\text{(i)}\quad \tan\frac{B+C}{2} = \frac{\cos\frac{b-c}{2}}{\cos\frac{b+c}{2}}\cot A/2,$$

$$\tan\frac{C+A}{2} = \frac{\cos\frac{c-a}{2}}{\cos\frac{c+a}{2}}\cot B/2,$$

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$$\tan\frac{A+B}{2} = \frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}\cot C/2.$$

$$\text{(ii)}\quad \tan\frac{B-C}{2} = \frac{\sin\frac{b-c}{2}}{\sin\frac{b+c}{2}}\cot A/2, \text{ etc.}$$

$$\text{(iii)}\quad \tan\frac{b+c}{2} = \frac{\cos\frac{B-C}{2}}{\cos\frac{B+C}{2}}\tan a/2, \text{ etc.}$$

$$\text{(iv)}\quad \tan\frac{b-c}{2} = \frac{\sin\frac{B-C}{2}}{\sin\frac{B+C}{2}}\tan a/2, \text{ etc.}$$

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6. $s = \frac{1}{2}(a + b + c)$ and $S = \frac{1}{2}(A + B + C)$

$$\text{(i)}\ \cos A/2 = \sqrt{\frac{\sin s \sin (s - a)}{\sin b \sin c}},\ \text{etc.}$$

$$\text{(ii)}\ \sin A/2 = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}},\ \text{etc.}$$

$$\text{(iii)}\ \tan A/2 = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}},\ \text{etc.}$$

$$\text{(iv)}\ \cos a/2 = \sqrt{\frac{\cos (S - B) \cos (S - C)}{\sin B \sin C}},\ \text{etc.}$$

$$\text{(v)}\ \sin a/2 = \sqrt{\frac{-\cos S \cos (S - A)}{\sin B \sin C}},\ \text{etc.}$$

$$\text{(vi)}\ \tan a/2 = \sqrt{\frac{-\cos S \cos (S - A)}{\cos (S - B) \cos (S - C)}},\ \text{etc.}$$

EXAMPLES XIX

1. Establish the formula for the area of a spherical triangle in terms of the angles.

   In a spherical triangle $\cos a = \cos b = \sqrt{3} \cos c = 1/\sqrt{3}$, show that the area of the triangle is $1/24$ of the surface of the sphere. *(II)*

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2. Prove that the area of a spherical triangle ABC on a sphere of radius R is $R^2(A + B + C - \pi)$.

   An equilateral triangle is drawn on a sphere so that a side of the triangle is equal to the radius of the sphere.

   Find the ratio of the area of the triangle to that of the whole surface of the sphere. *(II)*

3. Calculate the angles and sides of an equilateral triangle whose area is one fourth of the area of the sphere upon which it is drawn. *(II)*

4. Prove that, in a spherical triangle ABC,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

   Hence, or otherwise, prove that in an isosceles spherical triangle in which $b = c$, $\cos B = \cot b \tan a/2$, and calculate B if

$$a = 72°\ 30',\quad b = 46°\ 20'.$$ *(II)*

5. In a spherical triangle ABC, $b = c$. Prove that

$$\sin \tfrac{1}{2}a = \sin c \sin \tfrac{1}{2}A.$$

   Calculate the equal sides of an isosceles spherical triangle if the angle between them is 120° and the other side is 90°, and, if the radius of the sphere is R, show that the area of the spherical triangle is $\pi R^2/6$. *(II)*

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6. A spherical triangle ABC has a right angle at C; prove that

$$\text{(i) } \cos c = \cot A \cot B, \quad \text{(ii) } \cos a = \cos A / \sin B.$$

A ship sails from a point in N. latitude to a point on the equator, following a great circle course.

Its initial direction of motion is 30° W. of S. and its final direction is 24° W. of S. Find the length of its journey and the latitude of its starting point, taking the earth to be a sphere of radius 4,000 miles. *(II)*

7. ABC is a spherical triangle having the side AB equal to the side AC; prove that the median AD is at right angles to BC.

If in addition, $AD = BD = CD$, prove that

$$\cos BAC = -\tan^2 \tfrac{1}{2}AB. \qquad (II)$$

8. Prove that, in a spherical triangle ABC, right angled at C,

(i) $\cos c = \cos a \cos b$, (ii) $\sin A = \sin a / \sin c$,

(iii) $\cos A = \tan b / \tan c$.

Solve completely the spherical triangle in which $C = 90°$, $A = 60°$, $a = 45°$. *(II)*

9. Find a formula for the tangent of half an angle of a spherical triangle in terms of its sides.

Calculate the smallest angle of the triangle whose sides are 32° . 2′, 77° . 46′, and 72° . 46′. *(II)*

10. In a spherical triangle XYZ the angle XZY is a right angle. Prove that $\cos YXZ = \cos XY \tan XZ$.

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From the vertex A of the spherical triangle ABC a great circle is drawn to meet the side BC at right angles at the point P.

Prove that $\dfrac{\cos BAP}{\cos CAP} = \dfrac{\cot BA}{\cot CA}$. *(II)*

11. The most southerly latitude reached by the great circle joining a place P on the equator to a place Q in south latitude ϕ is λ.

The difference in longitude between P and Q is x. Prove that $\cot \lambda = \cot \phi \sin x$, deducing the result from formulae of the type $\cos a = \cos b \cos c + \sin b \sin c \cos A$, and the sine formulae. *(II)*

12. The most southerly latitude reached by the great circle joining a place P on the equator to a place Q in north latitude λ is ϕ.

Prove that the difference of longitude between P and Q is $\sin^{-1}(\tan \lambda \cot \phi)$, and find the angle between the meridian through Q and the great circle PQ. *(II)*

13. Two ports have the same latitude l and their longitudes differ by $2d$.

Prove that the length of the shortest course between them is $2R \sin^{-1}(\sin d \cos l)$, where R is the radius of the earth.

Find the greatest distance along a meridian between the shortest route and the parallel of latitude between the two points.

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14. A station A is in latitude 47° . 20′ N. and longitude 25° . 30′ E., and another station B is in latitude 23° . 12′ S. and longitude 78° . 40′ E.

Find the angle at which the great circle route from A to B cuts the equator, and the longitude of the point of intersection. *(II)*

15. The latitude and longitude of two points A and B are (73° . 30′ W., 40° N.), and (18° . 25′ E., 33° . 59′ S.).

Find the length of the great circle track from A to B (in nautical miles) and determine the longitude in which it crosses the equator. *(II)*

16. A ship sails along a great circle. At noon on three successive days the latitudes are observed to be λ_1, λ_2, λ_3. Whilst the distance sailed in each day is (in angular measure) δ. Show that

$$\cos\delta = \sin\tfrac{1}{2}(\lambda_1 + \lambda_3)\cos\tfrac{1}{2}(\lambda_1 - \lambda_3)/\sin\lambda_2, \qquad (II)$$

and express the total difference of longitude in terms of the three given latitudes. *(II)*

17. The sides of an equilateral triangle of area S are slightly bent so as to convert it into a spherical triangle on a sphere of radius r, large compared with the side of the triangle.

If $\pi/3 + \theta$ be an angle of the spherical triangle, show that approximately,

$$\theta = \frac{S}{3r^2} + \frac{1}{6\sqrt{3}}\frac{S^2}{r^4},$$

and that the area of the spherical triangle is $S + \dfrac{1}{2\sqrt{3}}\dfrac{S^2}{r^2}$. *(II)*

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Answers

CHAPTER I (page 7)

1. $3/5 \leqslant k \leqslant 5/3$; $\dfrac{-7-\sqrt{33}}{2} < x < \dfrac{-7+\sqrt{33}}{2}$.
2. $A = 2(ab_1 - a_1b)$, $B = ac_1 - a_1c$. $C = 2(bc_1 - b_1c)$.
5. Extreme values 0, -16.
7. $x = 0$, $y = 0$, $z = 0$, or $x = \dfrac{k}{a+1}$, $y = \dfrac{k}{b+1}$, $z = \dfrac{k}{c+1}$,
   where $k = \sqrt{(a+b+c+2-abc)}$.
11. $a+b+c = -p$, $bc+ca+ab = q$, $abc = -r$.
12. $x^4 - 5x^2 + 6 = 0$; $x = -3 \pm \sqrt{3}$, or $x = -3 \pm \sqrt{2}$.
16. Deduced area $= \sin\phi\,\sqrt{(abcd)}$ sq. units.

CHAPTER II (page 17)

3. 1·609. 4. $x = \frac{1}{10}$, $y = -\frac{9}{10}$, $z = \frac{9}{2}$.
5. $\dfrac{\sqrt{2} \pm i\sqrt{2}}{2}$, $\dfrac{-\sqrt{2} \pm i\sqrt{2}}{2}$; length $= 4a$.
6. $r^2 = 2\cos 2\theta$; $x + x^2 + \dfrac{x^3}{3} - \dfrac{x^5}{30}$.
7. Centre (3R, $-$4R), radius 5R;
   $\frac{1}{64}\{35\sin\theta - 21\sin 3\theta + 7\sin 5\theta - \sin 7\theta\}$.
9. (i) $z + |z|e^{i\theta}$, (ii) $z + |z|\dfrac{\sqrt{3}}{2}e^{i\theta}$; $z + |z|\dfrac{\sqrt{3}}{2}e^{i(n\pi/3+\pi/6)}$.
10. $\sqrt[3]{4}\left\{\dfrac{\sqrt{3}}{2} + i\tfrac{1}{2}\right\}$, $\sqrt[3]{4}\left\{-\dfrac{\sqrt{3}}{2} + i\,.\,\tfrac{1}{2}\right\}$, $\sqrt[3]{4}\{0 - i\}$.
13. $\dfrac{y}{\sin y} = \cosh x$.
14. $\cos(8k+1)\dfrac{\pi}{16} + i\sin(8k+1)\dfrac{\pi}{16}$, where $k = 0, 1, 2, 3$.
17. $\tan z = (e^{iz} - e^{-iz})/i(e^{iz} + e^{-iz})$.
18. (i) $\sin z = \dfrac{1}{2i}\left(e^{iz} - e^{-iz}\right)$, $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$.
19. $u = \{(x-1)\cos\alpha + y\sin\alpha\}\left\{\dfrac{x^2+y^2-2x+2}{(x-1)^2+y^2}\right\}$,
    $v = \{y\cos\alpha - (x-1)\sin\alpha\}\left\{\dfrac{x^2+y^2-2x}{(x-1)^2+y^2}\right\}$.

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21. Radius 1 ; Anticlockwise.

22. $x = \dfrac{c \tanh u \{1 + \tan^2 v\}}{1 + \tanh^2 u \tan^2 v}, \quad y = \dfrac{c \tan v \{1 - \tanh^2 u\}}{1 + \tanh^2 u \tan^2 v}.$

23. (i) 2π, (ii) $Q \equiv \frac{1}{2}\{-(2 + \sqrt{3}) + i(2\sqrt{3} - 1)\}$,
$R \equiv \frac{1}{2}\{-(2 - \sqrt{3}) - i(2\sqrt{3} + 1)\}$.

26. $z = -\frac{1}{2}\left\{i \cot \dfrac{k\pi}{8} + 1\right\}$, where $k = 1, 2, 3, \ldots, 7$.

CHAPTER III (page 26)

1. $\dfrac{1}{2(1-x)^2} - \dfrac{1}{2(1-x)} - \dfrac{x}{2(1+x^2)}$;

$\frac{1}{2}(2n^2 + n - 1)$ for n odd and $\dfrac{n}{2}(2n + 1)$ for n even.

2. $\dfrac{a(a-b)}{(1-ax)^2} - \dfrac{ab}{1-ax} + \dfrac{b^2}{1-bx}$; $(n+1)(a-b)a^{n+1} - a^{n+1}b + b^{n+2}$;
$|x| < |1/a|$ if $|a| > |b|$ or $|x| < |1/b|$ if $|b| > |a|$.

3. $\dfrac{a^2-1}{a(a-b)(1-at)} - \dfrac{1}{ab} + \dfrac{b^2-1}{b(b-a)(1-bt)}$, $a \neq b$;

$\dfrac{2}{a^2(1-at)} - \dfrac{1}{a^2} + \dfrac{a^2-1}{a^2(1-at)^2}$, if $b = a$.

4. $\dfrac{1}{t}\{1 - (1-t^2)^{\frac{1}{2}}\}$. 5. (ii) 2·00521, (iii) $\frac{28}{27} - \sqrt[3]{\frac{10}{9}}$. 6. (*b*) $\cos\theta$.

7. $S_n = \dfrac{n(3n+5)}{8(3n+1)(3n+4)}$, $S_\infty = \dfrac{1}{24}$, (i) $5e - 1$, (ii) $\dfrac{2(2-\cos\theta)}{3(5-4\cos\theta)}$.

9. When $x = 1$ sum $= \frac{1}{4}$.

10. $\log_e \dfrac{1+x}{1-x} = 2\{x + x^3/3 + x^5/5 + \ldots\}$, for $|x| < 1$;

$\tan^{-1} x = x - x^3/3 + x^5/5 - x^7/7 + \ldots$, for $|x| < 1$;

$y = e^{-2} \sum_{n=1}^{\infty} \dfrac{x^{2n-1}}{2n-1}$.

11. (*a*) (i) $\dfrac{n(n^2+6n+11)}{18(n+1)(n+2)(n+3)}$, (ii) $\cot\theta - \cot 2^n\theta$,
(*b*) $\frac{1}{2}\{e - e^{\cos\theta}\cos(\sin\theta)\}$.

12. (i) $\frac{1}{6}\left\{\dfrac{4}{n+1} + \frac{1}{6} - 5\left(\dfrac{1}{n+2} + \dfrac{1}{n+3}\right)\right\}$,
(ii) $\dfrac{(n+1)\sin n\theta - n\sin(n+1)\theta}{2(1-\cos\theta)}$.

13. $\dfrac{2(1-x^n)}{(1-x)^2} - \dfrac{\{1 + (2n-1)x^n\}}{1-x}$.

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14. (i) $\sin 3x = 3\sin x - 4\sin^3 x$; $S_n = \frac{1}{4}(3^n \sin\theta/3^n - \sin\theta)$,
$S_\infty = \frac{\theta}{4} - \frac{1}{4}\sin\theta$. (ii) $n\pi + \tan^{-1}\dfrac{b\sin C}{a - b\cos C}$.

CHAPTER IV (page 30)

1. (i) 2, (ii) $\frac{1}{2}$, (iii) $1/e$.

2. (i) -18, (ii) $\dfrac{na^2x^{n-1}}{\{x + \sqrt{(a^2 - x^2)}\}^{n+1}\sqrt{(a^2 - x^2)}}$,

(iii) $-\frac{3}{4}\operatorname{cosec}^3\dfrac{3\theta}{2}\sec\dfrac{\theta}{2}$. **3.** (i) (*a*) 1, (*b*) 0.

4. (i) $x^n e^{\cos x}\{n/x - \sin x\}$, (ii) $\dfrac{2 + x^2 + x^3}{2x(1+x)(1+x^2)}$; $A = -1$, $B = \frac{1}{2}$.

5. (i) $\dfrac{a(a^2 - x^2)}{(a^2 - ax + x^2)^{3/2}(a^2 + ax + x^2)^{\frac{1}{2}}}$,

(ii) $(\cosh x)^x\{x\tanh x + \log_e(\cosh x)\}$.

6. 0; the sum of the two functions is a constant.

7. (i) $\dfrac{3x^2 - x - 2}{2(1 - x^3)^{3/2}}$, (ii) $\dfrac{m\sec^2 x}{1 + m^2\tan^2 x}$,

(iii) $\dfrac{x}{a^2 + x^2}\sec\{\frac{1}{2}\log(a^2 + x^2)\}\tan\{\frac{1}{2}\log(a^2 + x^2)\}$.

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8. $2x\cos x^2$; (i) $\dfrac{2(x+y)\cos(x+y)^2}{1 - 2(x+y)\cos(x+y)^2}$, (ii) $\frac{2}{3}\sec^3 2\theta\operatorname{cosec}\theta$.

10. (i) $\dfrac{-2}{\sqrt{(1 - x^2)}}$, (ii) -1.

11. (*c*) $\dfrac{d^nz}{dx^n} = \alpha^{n-2}\left\{\alpha^2x^2\sin\left(\alpha x + \dfrac{n\pi}{2}\right) + 2\alpha nx\sin\left(\alpha x + \dfrac{n-1}{2}\pi\right)\right.$
$\left. + n(n-1)\sin\left(\alpha x + \dfrac{n-2}{2}\pi\right)\right\}$.

12. $(-1)^{n-1}2^{n-2}\dfrac{\{(n-1)!\}^2}{(2n-1)!}\dfrac{x^{2n+1}}{2n+1}$.

13. $y = ax + x^2 + \frac{1}{6}ax^3 + \frac{1}{3}x^4 + \frac{3}{40}ax^5 + \ldots$;
coefficient of x^{2p} is $2^{2p-2}\{(p-1)!\}^2/(2p-1)!\,p$,
coefficient of x^{2p+1} is $\{(2p)!\}^2a/2^{2p}(p!)^2(2p+1)!$

14. $(1^2 - m^2)(3^2 - m^2)\ldots\{(2m-1)^2 - m^2\}\dfrac{x^{2m+1}}{(2m+1)!}$

15. $b = -\frac{5}{4}$, $c = \frac{3}{8}$, $d = -\frac{1}{64}$.

16. $y = 1 + x + 7x^2 + 19x^3 + 145x^4 + 601x^5$.

19. $1 - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{12}x^4 + \frac{1}{3}x^5 - \frac{19}{72}x^6$.

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CHAPTER V (page 38)

1. $\tan^5\theta = \frac{1}{24}\left\{\frac{d^4}{d\theta^4}(\tan\theta) - 20\frac{d^2}{d\theta^2}(\tan\theta) + 24\tan\theta\right\}$;

$\frac{1}{4}\tan^4\theta - \frac{1}{2}\tan^2\theta + \log.(\sec\theta) + C.$

2. (i) $\log_e(\tan\theta/2) + \dfrac{2}{1+\tan\theta/2} + C.$

3. $\frac{1}{24}\log_e\dfrac{(x^2-9)(x+3)}{(x-1)^3} + C.$

4. $\frac{1}{12}\log_e\dfrac{(x+2)^4}{(1-x)(x+3)^3} + C$; $\frac{1}{2}C\sin^{-1}\dfrac{2x-c}{C} - \sqrt{(Cx-x^2)} + D.$

5. (i) $x\log_e x - x + C.$

6. (i) $3/2\log_e(x-1) + 7/6\log_e(x+1) - 16/3\log_e(2x-1) + C.$

(ii) $\log_e(\cos\theta + \sin\theta) + C.$

7. (i) $\frac{1}{4}\log_e\dfrac{(x-1)^2}{x^2+1} + \frac{1}{2}\tan^{-1}x + C$, (ii) $\log_e\left(1+\tan\dfrac{x}{2}\right) + C$;

$$\frac{1}{3}\left[\tan^{-1}x + \frac{x(2x^3+x^2-1)}{(1+x^2)^2}\right] + C.$$

8. (i) $2\theta - \log_e(1+\tan^2\theta/2) - 3\log_e(2+\tan\theta/2) + C.$

9. (a) $\frac{1}{3}\left\{\log_e(x^3+4x^2+6x) - 4\sqrt{2}\tan^{-1}\dfrac{x+2}{\sqrt{2}}\right\} + C.$

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10. (i) $\cosh^{-1}(2x-3) + C$; (ii) $\dfrac{x}{\sqrt{(1+x^2)}} + C$;

(iii) $\frac{1}{2}\left\{\sec x\tan x + \log_e\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)\right\} + C.$

11. (i) $C - \dfrac{1}{a^3}\left\{\dfrac{a}{x} + \tan^{-1}x/a\right\}$, (ii) $\sinh^{-1}(x+2) + C$;

(iii) $\frac{1}{4}x^2 - \dfrac{x}{4}\sin 2x - \frac{1}{8}\cos 2x + C.$

12. (i) $\pi = 3{\cdot}14$; (ii) $\frac{4}{9} = 0{\cdot}444$; (iii) $\frac{1}{15}\log_e 4 = 0{\cdot}0924$;

(iv) $\frac{3}{8} - \frac{19}{8}e^{-2} = 0{\cdot}0536.$

13. (i) $\pi\cos\dfrac{\pi}{n}\left[1 - \frac{1}{3}\cos^2\dfrac{\pi}{n}\right]$; (ii) $\log_e(3+2\sqrt{3}).$

14. (i) $\sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta + C$; (ii) $\frac{2}{15}(1+x)^{3/2}(3x-2)$;

(iii) $C - 2e^{-x/2}(x^2+4x+8)$; (iv) $\log_e(\tanh u/2) + C.$

15. (i) $0{\cdot}347$; (ii) $\dfrac{\pi}{4} - \frac{2}{3} = 0{\cdot}119$; (iii) $\dfrac{5\pi}{32} + \frac{1}{12} = 0{\cdot}574.$

16. (i) $\frac{1}{140} = 0{\cdot}00714$; (ii) $\frac{5}{12}\sqrt{2} = 0{\cdot}589$;

(iii) $\dfrac{\pi}{2} = 1{\cdot}57$; (iv) $\dfrac{\pi}{2} = 1{\cdot}57.$

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17. $\dfrac{1}{\sqrt{(A^2 - B^2)}} \cot^{-1} \dfrac{A^2 - B^2}{B + A \sin x} + C.$

18. (i) $\dfrac{2}{\sqrt{3}} \tan^{-1} \dfrac{2 \tan x + 1}{\sqrt{3}} + C$; (ii) $\dfrac{3\pi^2}{512}$.

19. (i) $\frac{1}{2} \log_e (x^2 + 1) - \log_e (x + 1) + C$; (ii) $\frac{8}{3} \log_e 2 - \frac{7}{9}$;
(iii) $\dfrac{\pi}{64} + \frac{1}{48}$. **20.** $\dfrac{\pi}{32}$.

22. $(n + 1) \displaystyle\int (x^2 + a^2)^{n/2}\, dx = x(x^2 + a^2)^{n/2} + na^2 \int (x^2 + a^2)^{\frac{n}{2}-1} dx.$

24. $W_{4,4,} = \dfrac{1}{16a^3} \tan^{-1} x/a + \dfrac{x}{16a^2(x^2 + a^2)} - \dfrac{x}{8(x^2 + a^2)^2} - \dfrac{x^3}{6(x^2 + a^2)^3}.$

25. $\dfrac{(2p - 1)!(2q - 1)!}{2^{2(p+q-1)}(p + q)!(p - 1)!(q - 1)!} \times \dfrac{\pi}{2}.$

26. $I_3 = x + \sin 2x + \frac{1}{2} \sin 4x$; $J_3 = 3x + 2 \sin 2x + \frac{1}{2} \sin 4x$.

27. $\dfrac{e^x}{85} \{5 \cos^3 x (\cos x + 4 \sin x) + 12 \cos x (\cos x + 2 \sin x) + 24\}.$

CHAPTER VI (page 46)

1. $x = \dfrac{\pi}{3} + \sqrt{3}\left(\dfrac{\alpha}{6} - \dfrac{\alpha^2}{72}\right).$ **3.** $1 + \dfrac{x^2}{6} + \dfrac{7x^4}{360}.$

4. $1 + \dfrac{x^2}{12} - \dfrac{x^4}{720}.$ **5.** P0538.

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6. (i) $\dfrac{x}{a} - \dfrac{x^3}{3a^3} + \dfrac{x^5}{5a^5} - \dfrac{x^7}{7x^7} + \ldots$, if $x^2 \leqslant a^2$;

(ii) $\dfrac{\pi}{2} - \dfrac{a}{x} + \dfrac{a^3}{3x^3} - \dfrac{a^5}{5x^5} + \ldots$, if $x^2 > a^2$ or $x = a$.

7. $\log_e (1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots;$

$\log_e \dfrac{1 + x}{1 - x} = 2\left(x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \ldots\right)$; $\log_e 11/9 = 0{\cdot}201$.

9. $\dfrac{d}{1 + \sqrt[3]{b/a}}.$ **10.** $68^\circ . 31'$ to base extended.

11. (i) Maximum $4/5$, minima 0 and -12; (ii) 152°.

12. (*a*) $\dfrac{-1}{(1 + x)\sqrt{x} \sin^{-1} \dfrac{1 - x}{1 + x}}$; (*b*) $2^c = 114^\circ . 35'$.

13. (*a*) $\dfrac{6x + 17}{2x + 3} + 3 \tan (x - 2).$ **14.** $AE = AD = \sqrt{\dfrac{AB . AC}{n}}.$

16. $4\sqrt{3} . a^2/9.$ **18.** $x = p\pi$ or $x = 2q\pi \pm \cos^{-1} \alpha/\beta.$

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19. OM $= (\sqrt{17} - 3)a$.

20. $x = \frac{1}{4}(9 + \sqrt{17})$ gives maximum and $x = \frac{1}{4}(9 - \sqrt{17})$ gives a minimum value of y.

21. The $+$ *ve* sign gives a maximum value and $-$ *ve* sign a minimum value of y.

22. Minimum value $2(n - m^3)$, maximum value $2(n + m^3)$.

CHAPTER VII (page 50)

1. $xt(3 + t^2) + 2y = 3t$.

2. $tx + y = 2at + at^3$; $x = 2at - \dfrac{l}{\sqrt{(1+t^2)}}$; $y = at^2 + \dfrac{lt}{\sqrt{(1+t^2)}}$.

4. $27ay^2 = 4(x - 2a)^3$. 5. $-4a$. 6. $\dfrac{125}{448}$; $\left(\dfrac{149}{448}, \dfrac{95}{112}\right)$.

7. $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$.

8. $x \tan t_0 + y = x_0 \tan t_0 + y_0$; $-2e^{t_0}$. 9. CP $= 9/4$.

14. $x^2 + y^2 = a^2$. 16. $a^8p^2 = r^{10}$. 18. $6prc^2 = r^4 + 8c^4$.

19. $r = ae^{\theta\frac{2}{3}}$. 20. Angle B $= \pi/2$.

28. (i) $w = xy(x + y)e^{x+y}$. 31. (ii) $-\dfrac{32}{21}$ in. per sec.

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CHAPTER VIII (page 58)

1. (i) $-(a + b + c)(b - c)(c - a)(a - b)$;

   (ii) $\theta = n\dfrac{\pi}{2} + (-1)^n\left(\dfrac{-\pi}{12}\right)$.

5. (i) $n(x + y + z)^3$.

6. (i) $2(a + b + c)(b - c)(c - a)(a - b)$; (ii) $x = 1$ (twice), or 2.

7. (ii) (*a*) $x^4(x-1)^3(x+1)$; (*b*) $(\alpha-\beta)(\gamma-\delta)(\beta-\delta)(\alpha-\gamma)(\beta-\gamma)(\delta-\alpha)$.

8. $(b - c)(c - a)(a - b)$; $-(b - c)(c - a)(a - b)^2$.

9. (i) $(x + y + z)(y - z)(z - x)(x - y)$;

   (ii) $\begin{vmatrix} 1 & 1 & 1 \\ 1/a & 1/b & 1/c \\ a^2 & b^2 & c^2 \end{vmatrix} = 0$; $5 : 3 : -8$.

10. $x = \dfrac{-(b + c + 1)(b - 1)(c - 1)}{a(a + b + c)(c - a)(a - b)}$;

    $y = \dfrac{-(c + a + 1)(a - 1)(c - 1)}{b(a + b + c)(b - c)(a - b)}$;

    $z = \dfrac{-(a + b + 1)(a - 1)(b - 1)}{c(a + b + c)(b - c)(c - a)}$.

11. (*a*) $x = 4$; (*b*) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = 0$;
$a = b$, or $b = c$, or $c = a$.

12. Eliminant $(b - q)^2 + a(b - q)(p - a) + b(p - a)^2 = 0$.

13. $\lambda = 1,\ x = -5,\ y = 1$; $\lambda = -1,\ x = -1/11,\ y = -15/11$; $\lambda = 12,\ x = 1/2,\ y = 1$.

CHAPTER IX (page 62)

2. One pair $3x - y - 1 = 0$, $3x + y - 14 = 0$.

3. Conditions are $h(p^2 - q^2) = (a - b)pq$, and $3a^2 + 4ab + 3b^2 = 4h^2$.

4. $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

5. $\dfrac{x^2 - y^2}{a - b} = \dfrac{xy}{h}$; $\dfrac{a_1 - b_1}{a - b} = \dfrac{h_1}{h}$; $\alpha\beta x^2 - (\alpha^2 + \beta^2)xy + \alpha\beta y^2 = 0$.

7. $(x - 3y)(3x + y) = 0$; $x^2 + y^2 - 2x - 4y = 0$.

9. $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$;
$(0,\ 0),\ \left(\dfrac{-ln}{l^2 + m^2},\ \dfrac{-mn}{l^2 + m^2}\right)$; $x^2 + y^2 + \lambda(mx - ly) + \dfrac{n}{m}y = 0$.

10. A.

11. $(x^2 + y^2)(a\cos^2\alpha - 2h\sin\alpha\cos\alpha + b\sin^2\alpha) + 2x(h\cos\alpha - b\sin\alpha)$
$+ 2y(h\sin\alpha - a\cos\alpha) + a + b - 1 = 0$.

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12. $(a + r\cos\theta,\ r\sin\theta)$. **13.** $2fy + c - g^2 = 0$; $\left(-g,\ \dfrac{g^2 - c}{f}\right)$.

14. $x^2 + y^2 - 5x + 5y = 0$.

16. $2(g - g_1)x + 2(f - f_1)y + c - c_1 = 0$;
$x^2 + y^2 - 2x - 4y - 13 = 0$.

17. $ly = x + al^2$; $(al^2,\ 2al)$. **18.** $ty = x + at^2$.

20. $tx + y = 2at + at^3$. **23.** $a = \pm\{\sqrt{(1 + 4t^2)}\}/(t^2 - 1)$.

24. $x^2 + y^2 - ax(1 + 3t^2) + ay(t^3 - 3t) + 3a^2t^2 = 0$.

25. $\tan^{-1}\dfrac{\pm 2\sqrt{(h^2 - ab)}}{a + b}$; $7y = 2x + 12a$, $7y + 2x + 12a = 0$.

27. $t_2 = -1/t_1$.

CHAPTER X (page 70)

1. $\alpha b^2(x - \alpha) + \beta a^2(y - b) = 0$. **6.** $b^2x^2 + a^2y^2 = 4a^2b^2$.

7. $b^4x^2 + a^4y^2 = a^2b^2(a^2 + b^2)$. **14.** $(x^2 - y^2)^2 = a^2(2x^2 - y^2)$.

19. $x = \dfrac{a^2h}{a^2 + b^2}$, $y = \dfrac{b^2k}{a^2 + b^2}$.

21. $\alpha b^2(x - \alpha) = \beta a^2(y - \beta)$; chord becomes tangent at $(\alpha,\ \beta)$.

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23. If $A \equiv (a \sec \alpha,\ b \tan \alpha)$ and the origin is the point of intersection of the fixed lines, the locus is the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{x}{a} \sec \alpha + \frac{y}{b} \tan \alpha = 0.$$

24. (i), (iii) Two different branches of the same hyperbola with major axis $(R - r)$ and eccentricity $e = AB/(R - r)$. (ii) Hyperbola with major axis $(R + r)$ and $e = AB/(R + r)$. Using mid-point of AB as origin the asymptotes are $y = \pm x \tan \alpha$, where for (i) and (iii) $\sec \alpha = AB/(R - r)$, and for (ii) $\sec \alpha = \dfrac{AB}{R + r}$.

CHAPTER XI (page 77)

1. $\left(\dfrac{ab^2}{a^2 + b^2},\ \dfrac{a^2 b}{a^2 + b^2}\right)$.

3. $(-1/4,\ -1/4)$.

4. $\left(\dfrac{fh - bg}{ab - h^2},\ \dfrac{gh - af}{ab - h^2}\right)$.

5. $y = -36/23$.

6. $e = 3$.

7. $ax^2 + 2hxy + by^2 + 2fy = 0$.

8. $x = x_1 \cos \theta - y_1 \sin \theta,\ y = x_1 \sin \theta + y_1 \cos \theta$; $\theta = \tan^{-1} 2/3$; hyperbola: transverse axis $\dfrac{2}{\sqrt{11}}$, conjugate axis $\sqrt{2}$.

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CHAPTER XII (page 81)

1. $(ax + by + cz + d)(a_1 l + b_1 m + c_1 n)$
$\quad - (a_1 x + b_1 y + c_1 z + d_1)(al + bm + cn) = 0.$

2. $a = 1,\ b = 1$; $\dfrac{x - 1}{1} = \dfrac{y}{1} = \dfrac{z - 1}{2}$.

3. $(a - d)\left(x - \dfrac{a + d}{2}\right) + (b - c)\left(y - \dfrac{b + c}{2}\right) + (c - f)\left(z - \dfrac{c + f}{2}\right) = 0.$

4. $(x - a)(mm_1 - m_1 n) + (y - b)(nl_1 - n_1 l) + (z - c)(lm_1 - l_1 m) = 0$;
$(4, 3, 2),\ 5x + 2y - 3z = 20.$

5. $Q \equiv \left(a,\ \dfrac{y_1 + z_1}{2},\ \dfrac{y_1 + z_1}{2}\right)$,

$R \equiv \left(\dfrac{3a + 2x_1 - y_1 - z_1}{3},\ \dfrac{3a - x_1 + 2y_1 - z_1}{3},\ \dfrac{3a - x_1 - y_1 + 2z_1}{3}\right)$,

$$6(x - a)^2 + 3(x - z)^2 = 2(x + y + z - 3a)^2.$$

6. $\dfrac{x}{1} = \dfrac{y - 2}{0} = \dfrac{z - 2}{1}$; direction cosines $\dfrac{4}{\sqrt{26}},\ \dfrac{-3}{\sqrt{26}},\ \dfrac{1}{\sqrt{26}}$.

9. $(0, 0, 0)$; $Q \equiv (-\frac{1}{2},\ 2,\ -\frac{3}{2})$.

11. $a\dfrac{\sqrt{6}}{3}$; $\dfrac{x}{1} = \dfrac{y}{1} = \dfrac{z - a}{2}$.

12. $3\sqrt{30}$; direction cosines $\frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}, \frac{-1}{\sqrt{30}}$.

13. $\sqrt{59}$; $46x + 55y - 13z = 384$, $28x - 5y - 31z = 126$.

16. $3x + 3y + 3z = 5a$.

CHAPTER XIII (page 88)

1. Radius $\sqrt{(u^2 + v^2 + w^2 - c)}$, centre (u, v, w); Polar plane $xx_1 + yy_1 + zz_1 - u(x + x_1) - v(y + y_1) - w(z + z_1) + c = 0$.

2. Sphere (i) $2\sqrt{2}(x^2 + y^2 + z^2) - 4a(x + y + z) + a^2\sqrt{2} = 0$,
Sphere (ii) $x^2 + y^2 + z^2 - 2a(x + y + z) + 2a^2 = 0$;
Centre $x = y = z = a(2 + \sqrt{2})/4$, radius $\frac{1}{4}a\sqrt{(12\sqrt{2} - 2)}$.

3. Centre $(-4, 8, 9)$, radius 4 units.

4. $x^2 + y^2 + z^2 - 4x - 6y + 2z + 5 = 0$; $x + 2y + 2z = 15$.

6. $\sqrt{\{p + q + 2\sqrt{(a^2 - p)(b^2 - q)}\}}$.

7. $9x^2 + 9y^2 + 9z^2 - 18ax - 36ay - 72az + 169a^2 = 0$; centre of circle $\left(\frac{169a}{189}, \frac{338a}{189}, \frac{676a}{189}\right)$, radius $\frac{26a}{9}\sqrt{\frac{5}{21}}$.

8. $x + y + z = 1 \pm \sqrt{3}$.

9. $(a - a_1)^2 + (b - b_1)^2 + (c - c_1)^2 = R^2 + R_1^2$.

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15. $\sin^{-1}\frac{167}{3\sqrt{3385}}$.

17. $\left\{\frac{\pm a^2}{\sqrt{(a^2 + b^2 + c^2)}}, \frac{\pm b^2}{\sqrt{(a^2 + b^2 + c^2)}}, \frac{\pm c^2}{\sqrt{(a^2 + b^2 + c^2)}}\right\}$.

18. $\frac{x - \alpha}{a\alpha} = \frac{y - \beta}{b\beta} = \frac{z - \gamma}{c\gamma}$.

19. Conic $\left.\begin{array}{l} 4x^2 + 3y^2 + z^2 - 8x = 0 \\ 4x + 3y + z = 0 \end{array}\right\}$, centre $(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$.

21. $afx + bgy + chz = 1$, $alx + bmy + cnz = 0$.

23. $\frac{a^2x^2}{A} + \frac{b^2y^2}{B} + \frac{c^2z^2}{C} = (ax^2 + by^2 + cz^2)^2$.

25. $xy_1z_1 + yz_1x_1 + zx_1y_1 = 3a^3$.

27. $\frac{x - x_1}{3x_1^2z_1 + 2y_1x_1} = \frac{y - y_1}{-3y_1^2} = \frac{z - z_1}{x_1^3}$.

30. $(1 + \lambda)^2x + (1 - \lambda)^2y + 16z = 4a$.

31. $2\alpha\beta(x - \alpha) + \alpha^2y - a^2z = 0$.

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CHAPTER XIV (page 97)

1. $\frac{45\pi}{2}(\pi - 2)$ cu. units.

8. (i) $\frac{8\pi a^2}{3}\{2\sqrt{2} - 1\}$ sq. units ;

(ii) $\frac{\pi a^2}{4}\{2\sqrt{2} + 5\cosh^{-1} 3\}$ sq. units.

9. Length $\frac{R\sqrt{3}}{3}$ units, volume $\frac{2\sqrt{3}}{27}\pi R^3$ cu. units.

10. $\frac{1}{6}\{22\pi + 15\sqrt{3}\}$ sq. units.

12. (i) $a(\cot\theta_1 - \cot\theta_2)$ units ; (ii) $2\pi a$ units.

15. $s = a[\theta - \tanh\frac{1}{2}\theta]$, $A = a^2[\theta - 2\tanh\frac{1}{2}\theta]$ sq. units.

17. $12a/5$. 18. $(\frac{8}{5}, \frac{4}{3})$.

19. Area 1/6 sq. unit ; centroid (6·4, 2·5).

20. Area 4/3 sq units ; $\bar{x} = 11/5$.

21. On radius (length a) of symmetry distant $\frac{a\sin\alpha}{\alpha}$ from centre, where 2α is the angle subtended at the centre by the arc ; for semi-circular area $4a/3\pi$ from base diameter on radius of symmetry ;
(i) *total* surface area $= 2\pi a\{2a + 2b + \pi b\}$ sq. units ;

(ii) volume $= \pi a^2\left\{\pi b + \frac{4a}{3}\right\}$ cu. units.

22. $\frac{28 - 4\pi}{7 - \pi/2}$ in. $= 2{\cdot}85$ in.

23. (i) $\frac{80\sqrt{3}}{27}$; (ii) $\frac{\pi}{2}$; centroid on radius (length a) of symmetry at a distance $3a/4$ from centre.

26. (*a*) $\frac{\pi a^2}{3}(3\pi d + 2a)$ cu. units, (*b*) $2\pi a\ \{d(\pi + 2) + a\}$ sq. units.

CHAPTER XV (page 103)

1. (i) $y^2 = \frac{\sec^2 x}{C - \frac{1}{2}\tan^4 x}$, (ii) $(x - y)^3 = C(x + y + 2)$.
2. (i) $y = 2x - 2 + Ce^{-x}$, (ii) $y^2 + xy - x^2 - 5y = C$.
3. (i) $2x^3 + 3x^2y + y^3 = C$; (ii) $y = Ce^x + \frac{1}{2}e^{-x}$.
4. (i) $y = Cx^2e^{1/x} - (1 + 2x + 2x^2)$;
(ii) $\alpha = 1$, $x^4 + 2x^3y + x^2y^2 + 3x^2y + 6xy^2 + 3y^3 = C$.

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5. $y^3 = x^3(3x^3 - 2)$.

6. (i) $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx}\tan x - y\sec^2 x = 0$;

(ii) $\tan^{-1}(x + y) = x + \dfrac{\pi}{4} - \frac{1}{2}$; $y = 0{\cdot}808$.

7. (i) $y = A\cos 2x + B\sin 2x + \dfrac{x}{4}\sin 2x + \frac{1}{4}x$;

(ii) $y^2 + 4xy - 4x^2 - 4y - 8x = C$;

(iii) $y = \dfrac{1}{(1 + 3x)^2}\{Ce^{3x} - (2 + 3x)\}$.

8. (i) $y = \dfrac{1}{\sin x(C + \cos x)}$; (ii) $y = C(x^2 - y^2)$;

(iii) $2e^x(\sin x - \cos x) + e^{-2y}(2y + 1) = C$;

(iv) $y = \dfrac{x + 1}{x}\{C - \frac{1}{2}e^{-x^2}\}$.

9. (i) $y = \dfrac{1 + x}{x(1 - x)}\left\{C - \dfrac{1}{1 + x}\right\}$;

(ii) $\log_e (x + y)^3(x - y) + \dfrac{2x}{x + y} = C$.

10. (i) $\log_e (x^2 + y^2) + 2\tan^{-1} y/x = C$; (ii) $y = -x\log_e Cx$;
(iii) $y^2 = Cx^{-3} - x$.

11. (i) $y = \frac{1}{2}e^x(1 + \tan x) + C\sec x$; (ii) $x^2 + y^2 = C(x + y)$.

12. $\left(\dfrac{dy}{dx}\right)^3 = \dfrac{y}{x}$; $x^{3/2} \pm y^{3/2} = C$.

13. $x^2 + y^2 = Cy$; $2xy\dfrac{dy}{dx} + x^2 - y^2 = 0$. 14. $r = C(1 + \cos\theta)$.

15. $(x^2 + y^2)(y + \sqrt{[x^2 + y^2]})^2 = 4(3 + 2\sqrt{2})x^2$.

16. $y^2 = x^2\{1 - 2\log_e x\}$.

17. Family is $y^2 = C\{C \pm 2x\}$; $y^2 = 4(1 + x)$, $y^2 = \frac{1}{2}(\frac{1}{2} - 2x)$.

18. $\dfrac{dy}{dx} = \dfrac{k \pm \sqrt{(k^2 - y^2)}}{y}$,

$$x = k\log_e y \pm \left\{\sqrt{(k^2 - y^2)} - k\log_e \frac{k \pm \sqrt{(k^2 - y^2)}}{y}\right\}.$$

19. $\dfrac{dy}{dx} = \dfrac{-y}{\sqrt{(x^2 + y^2)} - x}$, solution $\sqrt{(x^2 + y^2)} + x = C$; orthogonal system is $\sqrt{(x^2 + y^2)} - x = C_1$.

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CHAPTER XVI (page 107)

1. (i) $y = e^x\{A \cos(\sqrt{3}\,.\,x) + B \sin(x\sqrt{3}) - \frac{e^x}{6}\sin 3x$;

   (ii) $y = (A + B \log_e x)x^3 + \frac{1}{27}(3 \log_e x + 2) + x^2$.

2. (i) $y = e^{-x}\{A \cos x + B \sin x + x^3 - 6x\}$;

   (ii) general integral $y = Cx^2 + \frac{x}{C}$, singular solution $y^2 = 4x^3$.

3. (i) $y = e^{bx/a}\left\{A \cos\left(\frac{bx\sqrt{3}}{a}\right) + B \sin\left(\frac{bx\sqrt{3}}{a}\right)\right\} + b\left\{x^2 + \frac{a}{b}x\right\}$;

   (ii) general integral $y = Cx - \log C$; singular solution $y = 1 + \log_e x$.

4. (i) $y = (A + Bx + x^2/2)e^{2x}$;

   (ii) $y = x^{\frac{1}{2}}\left\{A \cos\left(\frac{\sqrt{3}}{2}\log_e x\right) + B \sin\left(\frac{\sqrt{3}}{2}\log_e x\right)\right\} + \frac{x^2}{3}$.

5. $y = e^{-2\sqrt{x}}\left\{A - \frac{5\sqrt{x} + 1}{25}\right\} + Be^{3\sqrt{x}}$.

6. $y = x^{-2}\left\{A + Be^{-4x} + \frac{4 \sin x - \cos x}{17}\right\}$.

7. (i) $y = e^{3x}(A + x/12) + e^{-3x}(B - x/12) - \frac{1}{81}(9x^2 + 2)$;

   (ii) $y = Ae^{3x} + Be^{-2x} + \frac{e^{3x}}{250}(25x^2 - 10x + 2)$.

8. (i) $y = Ae^{4x} + Be^{-x/2} - \frac{1}{7}e^{3x}$; (ii) $y = \frac{2}{3}x^3 + e^x + Ax + B$.

9. (i) $y = A \cos 3x + B \sin 3x + \frac{1}{81}(9x^2 + 9x + 7)$;
   (ii) $y = A \sec^2(x + B)$.

10. (i) $y = e^{-x}(A + \frac{1}{2}\sin x - \frac{1}{2}\cos x) + Be^{-2x}$;
    (ii) $y = 3(\tan x - 1) + Ae^{-\tan x}$.

11. $\frac{d^2z}{dx^2} + z = 0$; general solution $y = x^{-\frac{1}{2}}(A \cos x + B \sin x)$;

    particular solution $y = -\sqrt{\left(\frac{\pi}{x}\right)}\,.\cos x$.

12. (i) $y = e^x(A + Bx - \cos x)$; (ii) $y = e^z(A + z/4) + e^{-z}(B - z/4)$, where $z = \cosh^{-1} x$.

13. $\frac{d^2x}{dy^2} - 3\frac{dx}{dy} + 2x = y$; $x = Ae^y + Be^{2y} + \frac{1}{4}(2y + 3)$.

15. $y = A \cos 4x + B \sin 4x + C \cos 3x + D \sin 3x - 2$;
    $z = A/3 \cos 4x + B/3 \sin 4x - 2\,C \cos 3x - 2\,D \sin 3x$.

16. $x = Ae^t + Be^{-t} + Ce^{2t}$; $y = 2Ae^t - 2Be^{-t} + \frac{5}{2}Ce^{2t}$;

17. $x = Ae^z + Be^{2z} + Ce^{3z} + \frac{1}{36}(6z + 11)$;
    $y = -\{9Ae^z + 12Be^{2z} + 17Ce^{3z}\} + \frac{2}{9}(6z + 11)$.

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CHAPTER XVII (page 112)

1. $7\pi a^5/2$. 2. (a) $a4/8$; (b) $a^2b^2/24$.

3. $\displaystyle\int_0^a dx \int_0^a \frac{xdy}{\sqrt{(a^2-x^2)(a-y)(y-x)}} = \pi a.$

4. $\displaystyle\int_0^a dy \int_y^a \frac{dx}{\{(a-x)(x-y)(a-y)(a+y)\}^{\frac{1}{2}}} = \frac{\pi^2}{2}.$

5. (i) $\frac{1}{6}a^3 \sinh^{-1}\frac{b}{a} + \frac{ab}{6}\sqrt{(a^2+b^2)}$; (ii) $\frac{a^3}{3}\{\sqrt{2} + \log_e(1+\sqrt{2})\}$.

6. $\displaystyle\int_0^{\pi/2} dx \int_x^{\pi/2} \cos 2y \sqrt{(1-k^2\sin^2 x)}dy = \frac{1}{3k^2}[(1-k^2)^{\frac{3}{2}} - 1].$

7. $\displaystyle\int_0^a dy \int_0^y (x^2+y^2)dx = \tfrac{1}{3}a^4.$

8. $\displaystyle\int_{-a}^{a} \int_0^{\sqrt{(x^2+y^2)}} x\,dy\,dx = \tfrac{2}{3}a^3$ cu. units.

9. $\frac{1024}{15}$ cu. units. 11. $\dfrac{15\pi a^3}{16}$ cu. units. 12. $4\pi^2 ac$ sq. units.

13. (i) $\displaystyle\int_0^{2\pi} \int_0^{a(1+\cos\theta)} r dr\, d\theta = 3\pi a^2/2$ sq. units;

(ii) (a) $\displaystyle\int_0^{2\pi} \int_0^{a(1+\cos\theta)} r^2 \cos\theta\, dr\, d\theta$;

(ii) (b) $\displaystyle\int_0^{2\pi} \int_0^{a(1+\cos\theta)} r^3 \cos^2\theta\, dr\, d\theta.$

Centroid $r = 5a/6$, $\theta = 0$; radius of gyration $a\sqrt{\frac{35}{48}}$.

CHAPTER XVIII (page 115)

2. $\displaystyle\frac{2c}{\pi}\left\{\frac{\alpha}{2} + \sum_{n=1}^{\infty} \frac{1}{n}\sin n\alpha \cos nx\right\}.$

4. $\displaystyle -\frac{1}{\pi} + \frac{1}{2}\cos x - \frac{2}{3\pi}\cos 2x + \frac{2}{15\pi}\cos 4x + \ldots$

5. $\dfrac{\cos p\pi}{4p^2}\cos 2px.$ 6. $\displaystyle\frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\cos(2n+1)x}{(2n+1)^2}.$

8. $\displaystyle\frac{8}{\pi}\left\{\frac{1}{1^3}\sin x + \frac{1}{3^3}\sin 3x + \frac{1}{5^3}\sin 5x + \ldots + \frac{1}{(2n-1)^3}\sin(2n-1)x + \ldots\right\}.$

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9. $\frac{2}{\pi}\left\{\sin x - \sin 2x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x - \frac{1}{3}\sin 6x + \ldots\right\}$.

10. $\frac{4}{\pi}\left\{\frac{1}{2}\sin x - \frac{1}{3}\sin 3x + \frac{1}{10}\sin 5x + \frac{1}{14}\sin 7x + \ldots\right\}$.

12. $\frac{2\sqrt{3}}{3}\left\{\frac{1}{1}\cos x - \frac{1}{5}\cos 5x + \frac{1}{7}\cos 7x - \frac{1}{11}\cos 11x + \ldots\right\}$.

13. $$x(x-1)(x-2) = \frac{12}{\pi^3}\left\{\frac{1}{1^3}\sin \pi x + \frac{1}{2^3}\sin 2\pi x + \frac{1}{3^3}\sin 3\pi x + \ldots + \frac{1}{n^3}\sin n\pi x + \ldots\right\}.$$

CHAPTER XIX (page 119)

2. 1 : 25·3. 3. $A = B = C = 120°$, $a = b = c = 109°\,.\,32'$.
4. 45° . 35′. 5. $b = c = 54°\,.\,44'$. 6. 2, 761 ml., 35° . 34′.
8. $b = 35°\,.\,16'$; $c = 54°\,.\,44'$; $B = 45°$. 9. 32° . 44′.
12. $\tan^{-1}\left(\frac{\tan x}{\sin \lambda}\right)$, where $x = \sin^{-1}(\tan \lambda \cot \phi)$.
13. $R\left\{\left(\frac{\pi}{2} - l\right) - \tan^{-1}(\cos d \cot l)\right\}$. 14. 64° . 24′; 59° . 59′ E.
15. Arc AB = 6,741 nautical miles; 21° . 4′ W.
16. $\cos^{-1}\frac{2\cos^2\lambda_2(\sin\lambda_1\sin\lambda_3 + 1) - \cos^2\lambda_1 - \cos^2\lambda_3}{2\cos\lambda_1\sin^2\lambda_2\cos\lambda_3}$.

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